Steiner Triple Systems of Order 19 and 21 with Subsystems of Order 7

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Dedicated to Jennifer Seberry on the occasion of her sixtieth birthday

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Abstract

Steiner triple systems (STSs) with subsystems of order 7 are classified. For order 19, this classification is complete, but for order 21 it is restricted to Wilson-type systems, which contain three subsystems of order 7 on disjoint point sets. The classified STSs of order 21 are tested for resolvability; none of them is doubly resolvable.

Key words: Classification; Doubly resolvable design; Steiner triple system; Subsystem

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1 Introduction

A Steiner triple system of order v, briefly STS(v), is a 2-$(v, 3, 1)$ design, that is, a collection of triples (called *blocks*) out of a v-set (of *points*), such that every pair of distinct points occurs in exactly one triple. See [7,8] for surveys of Steiner triple systems.

It is well known that an STS(v) exists if and only if either $v \equiv 1 \pmod 6$ or $v \equiv 3 \pmod 6$. Two STS(v) are said to be *isomorphic* if there exists a bijection between the point sets that maps blocks onto blocks. The problem of classifying nonisomorphic Steiner triple systems has been settled for all admissible orders $v \leq 19$, see [11].

In this paper we study STS(19) and STS(21) with subsystems of order 7.

For $v = 19$ we employ in Section 2 a modification of the algorithm used in [11] to obtain a complete classification of the STS(19) with a subsystem of order 7. This data combined with the earlier classification [15] of the STS(19) with a subsystem of order 9 allows us to determine the subsystem structure of all the STS(19).

For $v = 21$ a complete classification is currently out of reach, but various classification results on STS(21) with additional properties exist [10,12,13,16]. In the current work, the Wilson-type STS(21) are classified. These are STS(21) that contain three subsystems of order 7 on disjoint point sets. Some preliminary results of this work can be found in [17–19]. Two alternative approaches for classifying and counting the Wilson-type triple systems of order 21 are considered in Section 3. The paper is concluded in Section 4 by discussing Kirkman triple systems KTS(21), which are resolutions of STS(21). None of the classified STS(21) is doubly resolvable.

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2 Steiner Triple Systems of Order 19 with a Subsystem of Order 7

Since the subsystem structure of the Steiner triple systems of order 19 was not examined during the extensive classification in [11], we perform a classification of all the STS(19) with nontrivial subsystems here. Instead of just rerunning the original program (which would require at least two years of CPU time on the computers used in the original study), we show a faster, alternative way of classifying these particular systems directly. The direct classification consumed 29 hours of CPU time on a 1.7-GHz PC.

Since the present algorithm is only a minor modification of the algorithm in [11], which is too extensive to be repeated here, we refer the reader to [11] and only outline the necessary modifications.

2.1 Modifications to the Original Algorithm

The classification in [11] starts from (all nonisomorphic) seeds consisting of exactly those blocks that intersect one particular block. Since all blocks of an STS(7) intersect pairwise, it now suffices to classify STS(19) starting from seeds in which an STS(7) is embedded. Out of the 14648 seeds, only 157 have this property.

Also the isomorph rejection part of the original algorithm has to be modified. Informally, we have to modify the parent test so that the parent seed associated with each STS(19) contains an embedded STS(7) if (and only if) the STS(19) contains an STS(7). This modification can be carried out by replacing the Pasch configuration vertex invariant Λ_P in [11] with a new vertex invariant Λ_S based on both the Pasch configurations and the STS(7) that occur in an STS(19).

The new invariant Λ_S is computed as follows. For each block B in the STS(19), we compute $S(B)$—the number of STS(7) in which B occurs—and $P(B)$—the number of Pasch configurations in which B occurs. Then, we form an ordered partition of the blocks so that two blocks B, B' are in the same cell of the partition if and only if both $S(B) = S(B')$ and $P(B) = P(B')$. The cells of the partition are ordered first by the value $S(B)$ (largest first, decreasing order) and then within cells of equal $S(B)$ by the value $P(B)$ (largest first, decreasing order). It is easily checked (cf. [11, Theorem 2.10]) that this ordered partition defines a vertex invariant.

The correctness of the modified algorithm can be seen as follows. Let $\mathcal{B} = \{B_1, \ldots, B_{57}\}$ be an STS(19) and let G be the associated block graph (cf. [11]). In the modified algorithm, we perform the parent test in [11] using the new invariant Λ_S; that is, in the notation of [11] we have $\sigma_G = \sigma_{G,\Lambda_S(G)}$ instead of $\sigma_G = \sigma_{G,\Lambda_P(G)}$. This modifies σ_G so that $p := \sigma_G^{-1}(1)$ gives a block B_p that occurs in the maximum number of STS(7) in the STS(19). In particular, if the STS(19) contains at least one STS(7), then B_p defines a seed that has at least one embedded STS(7). It is now straightforward to check by retracing the proof of [11, Theorem 3.3] that a representative from each isomorphism class of STS(19) with at least one STS(7) is accepted in the parent test when the main search is restricted to seeds that contain an embedded STS(7).

Our implementation of the invariant Λ_S combines the search for Pasch configurations and STS(7). This is convenient because an STS(7) contains seven Pasch configurations (for any point p in an STS(7), take the four blocks not incident with p to obtain a Pasch configuration). Thus, whenever a Pasch configuration is found, we search for three blocks such that (a) the blocks extend the Pasch configuration to an STS(7); and (b) the point common to the three blocks must—with respect to some prescribed order—be larger than any of the points in the Pasch configuration. It is easy to see that in this way each STS(7) is counted exactly once. Another useful observation (cf. [11, Theorem 3.6]) is that an STS(19) can be rejected in the parent test immediately if for some i either $S(B_1) < S(B_i)$, or $S(B_1) = S(B_i)$ and $P(B_1) < P(B_i)$.

We remark that the complexity of finding subdesigns is discussed in [3], where a few practical algorithms can be found as well.

2.2 Classification Results

There are 86701547 nonisomorphic STS(19) that contain at least one subsystem of order 7. Among these there exist 13673 nonisomorphic STS(19) that contain also a subsystem of order 9. (This result is inconsistent with the number 13529 reported in [15], see the following section for analysis.) Combined with the earlier classification of the 284457 STS(19) with a subsystem of order 9 in [15], the subsystem structure of all the STS(19) is now known. In total, there are 86972331 STS(19) that have a nontrivial subsystem. Thus, out of the 11084874829 nonisomorphic STS(19) [11], the majority (10997902498) is subsystem-free. This data is summarized in Table 1.

Since listing here all the STS(19) with nontrivial subsystems is impossible, we tabulate only the number of occurrences of certain properties of interest such as the number of subsystems. Anyone with further interest in these designs is encouraged to contact the first author for electronic listings of the designs.

Table 2 gives more detailed information on the types and number of subsystems that can occur in the STS(19). The columns "#7" and "#9" give the number of STS(7) and STS(9) that can occur in an STS(19). Column "Nd"

Table 1
Classification of STS(19) by subsystem type

		Nd	Ref
+	Subsystem-free	10997902498	
+	Subsystem of order 7	86701547	
+	Subsystem of order 9	284457	[15]
−	Subsystems of both orders 7,9	13673	
	Total STS(19)	11084874829	[11]

Table 2
Number of subsystems in STS(19)

#7	#9	Nd
0	0	10997902498
0	1	270784
1	0	86101058
1	1	12956
2	0	572471
2	1	641
3	0	11819
3	1	45
4	0	2449
4	1	25
6	0	75
6	1	5
12	0	2
12	1	1
Total		11084874829

lists the number of nonisomorphic STS(19) with a given frequency of subsystems. Table 3 extends Table 2 by including also the full automorphism group order. Table 4 gives information on the number of Pasch configurations (column "#P") in relation to the number of subsystems of order 7.

Table 3
Automorphism group order and subsystems in STS(19)

|Aut|	#7	#9	Nd	|Aut|	#7	#9	Nd
1	1	0	86033483	6	1	0	88
1	1	1	12435	6	1	1	12
1	2	0	555709	6	3	0	45
1	2	1	479	6	3	1	7
1	3	0	10742	6	4	0	9
1	3	1	17	6	4	1	1
1	4	0	1384	6	6	0	4
1	4	1	2	6	6	1	2
1	6	0	6	8	4	0	80
2	1	0	66643	8	4	1	1
2	1	1	509	8	6	0	8
2	2	0	15854	12	1	0	9
2	2	1	150	12	4	0	13
2	3	0	933	12	4	1	1
2	3	1	3	12	6	0	7
2	4	0	692	12	6	1	3
2	4	1	11	16	4	0	10
2	6	0	24	16	4	1	3
3	1	0	462	18	3	0	6
3	3	0	92	18	3	1	4
3	3	1	13	24	4	0	11
3	4	0	12	32	4	0	3
4	1	0	373	54	3	0	1
4	2	0	908	54	3	1	1
4	2	1	12	96	4	0	1
4	4	0	234	108	12	0	1
4	4	1	6	144	12	0	1
4	6	0	26	432	12	1	1

Table 4
Pasch configurations in STS(19) with a subsystem of order 7

#7	#P	Nd	#7	#P	Nd	#7	#P	Nd
1	7	11	1	36	357431	2	23	3549
1	8	71	1	37	226167	2	24	7295
1	9	576	1	38	140723	2	25	11723
1	10	3235	1	39	83431	2	26	18701
1	11	13629	1	40	50660	2	27	24989
1	12	45036	1	41	28168	2	28	33600
1	13	128510	1	42	16637	2	29	39055
1	14	309026	1	43	8695	2	30	46121
1	15	652691	1	44	5281	2	31	47072
1	16	1215665	1	45	2553	2	32	49797
1	17	2047062	1	46	1481	2	33	46420
1	18	3114526	1	47	755	2	34	45538
1	19	4339916	1	48	434	2	35	38982
1	20	5573414	1	49	165	2	36	35971
1	21	6656369	1	50	127	2	37	27881
1	22	7437929	1	51	66	2	38	24876
1	23	7817956	1	52	31	2	39	17941
1	24	7782173	1	53	12	2	40	15572
1	25	7365339	1	54	10	2	41	10349
1	26	6665790	1	55	3	2	42	8364
1	27	5783030	1	56	3	2	43	5046
1	28	4828656	1	58	1	2	44	4357
1	29	3889710	1	60	1	2	45	2324
1	30	3030464	2	17	2	2	46	1988
1	31	2278675	2	18	16	2	47	933
1	32	1666095	2	19	51	2	48	938
1	33	1182401	2	20	256	2	49	305
1	34	816838	2	21	669	2	50	342
1	35	546386	2	22	1704	2	51	105

Table 4 (cont.): Pasch configurations in STS(19) with a subsystem of order 7

#7	#P	Nd	#7	#P	Nd	#7	#P	Nd
2	52	138	3	46	509	4	46	301
2	53	29	3	47	175	4	47	134
2	54	59	3	48	401	4	48	235
2	55	7	3	49	128	4	49	77
2	56	26	3	50	243	4	50	248
2	57	2	3	51	107	4	51	68
2	58	13	3	52	166	4	52	142
2	60	4	3	53	12	4	53	22
2	62	2	3	54	66	4	54	129
3	25	2	3	55	11	4	55	42
3	27	6	3	56	39	4	56	67
3	28	6	3	57	9	4	57	13
3	29	15	3	58	23	4	58	50
3	30	57	3	60	8	4	59	6
3	31	185	3	62	3	4	60	20
3	32	294	3	64	1	4	62	21
3	33	597	4	28	2	4	64	2
3	34	779	4	30	3	4	66	5
3	35	830	4	34	8	4	70	1
3	36	768	4	36	26	6	50	3
3	37	920	4	37	5	6	54	12
3	38	1014	4	38	53	6	58	17
3	39	781	4	39	3	6	60	8
3	40	870	4	40	113	6	62	21
3	41	584	4	41	35	6	66	13
3	42	739	4	42	199	6	70	4
3	43	473	4	43	79	6	78	2
3	44	652	4	44	249	12	84	3
3	45	391	4	45	116			

2.3 Correctness

We perform three tests to ensure that the classification results reported in the previous section are correct.

First, we count in two different ways in all labeled STS(19) the total number of blocks that occur in at least one subsystem of order 7. This test is a minor modification of the method used to check the correctness of the classification in [11]. Denote by $N_7(19)$ the number of nonisomorphic STS(19) with at least one subsystem of order 7 and let $X_1, \ldots, X_{N_7(19)}$ be representatives of these isomorphism classes of STS(19). Denote by z_i the number of blocks in X_i that occur in at least one STS(7). Let $S_1, \ldots, S_{157}$ be the seeds that contain at least one STS(7). Denote by M_j the number of labeled STS(19) that contain S_j.

By the orbit-stabilizer theorem, the total number of blocks in all labeled STS(19) that occur in at least one subsystem of order 7 is

$$\sum_{i=1}^{N_7(19)} \frac{19! \cdot z_i}{|\mathrm{Aut}(X_i)|} = \sum_{j=1}^{157} \frac{19! \cdot M_j}{|\mathrm{Aut}(S_j)|}. \tag{1}$$

The left-hand side of (1) is straightforward to evaluate by storing z_i and $|\mathrm{Aut}(X_i)|$ for each X_i encountered in the search. Similarly, the right hand side of (1) can be evaluated by storing for each seed S_j the number of extensions M_j of S_j to an STS(19) found by the exact cover algorithm (see [11]). Both counting approaches give the same result 74224504030945473773568000. This gives us confidence that the classification is correct.

As a second test, we follow [15] and perform an independent classification of the STS(19) with a subsystem of order 9 by combining the $9!/432 = 840$ labeled STS(9) in all possible ways with the 396 nonisomorphic 1-factorizations of the complete graph K_{10} (which are listed in, e.g., [1]) and rejecting isomorphs.

In agreement with [15], we obtain 284457 nonisomorphic STS(19) with a subsystem of order 9. Among these we find 13673 nonisomorphic STS(19) that also contain at least one subsystem of order 7, which is in agreement with the result obtained from our order 7 subsystem classification, but which is inconsistent with the number 13529 reported in [15]. We used two independent software implementations to verify that our 13673 STS(19) are indeed pairwise nonisomorphic and that they have subsystems of both orders (7 and 9). Based on this we believe that the result in [15] must be in error.

As a third test, we perform an independent enumeration of the Wilson-type STS(19) that have a nontrivial automorphism. A *Wilson-type* STS(19) contains three STS(7) that intersect pairwise in one common point [20]. The point sets of the three STS(7), excluding the common point, form a partition of the remaining 18 points. Consequently, the blocks not in the three STS(7) form a transversal design TD$(3, 6)$, which in turn can be viewed as a latin square of side 6.

The techniques used here are similar to those described later in Section 3.2 to which we refer the reader for details. The admissible automorphisms of prime order, called *basic automorphisms*, for a Wilson-type STS(19) are as follows:

(1) an automorphism of type $1^3 2^8 s$ with 9 fixed blocks,
(2) an automorphism of type $1^7 2^6$ with 13 fixed blocks,
(3) an automorphism of type $1^7 2^6 s$ with 13 fixed blocks,
(4) an automorphism of type $1^1 3^6$ with 3 fixed blocks,
(5) an automorphism of type $1^1 3^6 s$ with 0, 3, or 6 fixed blocks,
(6) an automorphism of type $1^7 3^4$ with 9 fixed blocks.

The ’s’ in the above notations of the automorphism type means that the automorphism transforms some of the STS(7) into one another, while in the remaining cases each of the three STS(7) is transformed into itself.

Searching for Wilson-type STS(19) admitting such automorphisms, we obtain the results in Table 5. Note that although we know the total number of (labelled) STS(19) with one Wilson-type configuration individualized, cf. [20], we cannot deduce the number of nonisomorphic Wilson-type STS(19) by classifying only those that possess nontrivial automorphisms (see Section 3, where this approach is used to enumerate Wilson-type STS(21)). Namely, such a counting argument works only if we know the number of sets of three particular STS(7) in the designs with trivial automorphism group.

A search for Wilson-type designs among the classified STS(19) with a subsystem of order 7 reveals 10489 such designs. All of these designs with a trivial automorphism group have just one set of three STS(7) making it a Wilson-type STS(19). From this observation, we can use the classification of designs with nontrivial automorphism groups to get the count in an alternative—but not completely independent—way.

3 Wilson-Type Steiner Triple Systems of Order 21

Wilson-type Steiner triple systems are named after Wilson, who considered these to obtain a general lower bound on the number of nonisomorphic Steiner triple systems [20]. Several constructions are used in [20]. For STS(21), Wilson considers Steiner triple systems that consist of three STS(7) with disjoint point

Table 5
Wilson-type STS(19) with nontrivial automorphisms

$\lvert\mathrm{Aut}\rvert$	$1^3 2^8 s$	$1^7 2^6$	$1^7 2^6 s$	$1^1 3^6 s$	$1^1 3^6$	$1^7 3^4$	Nd
2	126	230	482				838
3				41		64	105
4	98	137	74				137
6	16	12	40	22	8	38	68
8	8	8	8				8
12	18	21	18	11		10	21
18	2		8	2	10	8	10
54			2	2	2	2	2
108	1	1	1	1	1	1	1
144	1	1	1	1	1		1
432	1	1	1	1	1	1	1
Total	271	411	635	81	23	124	1192

sets, and shows that the number of these is at least 2160980. We shall now look at some properties of such designs.

Lemma 1 *A Wilson-type* STS(21) *contains exactly three* STS(7).

Proof. Consider an arbitrary Wilson-type STS(21) and let X_1, X_2, X_3 be subsystems of order 7 with pairwise disjoint point sets. If an STS(v) contains more than one subsystem, any two subsystems intersect in a subsystem [8, Lemma 6.45]. Therefore any subsystem X of order 7 must either (i) be one of X_1, X_2, X_3; or (ii) intersect the point sets of X_1, X_2, X_3 in 1, 3, and 3 points (in some order). But, in case (ii), the intersections in 3 points correspond to two nonintersecting blocks, and this is impossible since the blocks in an STS(7) have pairwise nonempty intersection. $\square$

The blocks not in the three STS(7) form a transversal design TD$(3, 7)$, which in turn can be viewed as a latin square of side 7. This leads to one possible approach for classifying the Wilson-type STS(21). This approach is considered in the next subsection.

In an alternative approach, we only enumerate—that is, count—the number of Wilson-type STS(21). From [20] we know that the total number of (labeled) such systems with given point sets P_i, $i \in \{1, 2, 3\}$, for the three STS(7) is $N = 30^3 \cdot 7! \cdot 6! \cdot 16942080$. Let N_i denote the number of systems with full automorphism group of order i. Then, by the orbit-stabilizer theorem,

$$N = 3! \cdot (7!)^3 \cdot \sum_i \frac{N_i}{i}. \qquad (2)$$

If we know N_i for all $i \neq 1$, then this equation can be solved for N_1, and $\sum_i N_i$ gives the desired count. The counts N_i for $i \geq 2$ are obtained by prescribing automorphism groups and classifying corresponding objects.

3.1 Combining Latin Squares and Three STS(7)

As we have seen, a Wilson-type STS(21) consists of blocks forming three STS(7) and a TD$(3, 7)$. One may then get at the classification problem by combining all such objects in all possible ways. In fact, we need not combine all objects, but for one part—three STS(7) or TD$(3, 7)$—it suffices to consider one representative from each isomorphism class.

An STS(7) is unique up to isomorphism and has a full automorphism group of order 168, so by the orbit-stabilizer theorem there are $7!/168 = 30$ distinct STS(7). There are 147 TD(3,7) up to isomorphism—in other words, there are 147 *main classes* of latin squares of side 7; see [4] for a list of these. The total number of TD(3,7), or *reduced* latin squares, is 16942080. Thereby, in the two alternative approaches, depending on what part is fixed, there are $147 \cdot 30^3 = 3969000$ or 16942080 systems, respectively, to consider. Because the former count is smaller, and the objects involved are easily available, we chose that approach.

Since the number of nonisomorphic Wilson-type STS(21) are only in the millions, isomorph rejection can be carried out in a brute-force manner by comparing and saving certificates for the constructed designs. As in [11], the certificates can be calculated by the graph canonical labeling program *nauty* [14] for the block graphs of the STS(21), utilizing an invariant based on Pasch configurations.

The classification outlined above consumed just under 4 hours of CPU time on a 1.7-GHz PC.

3.2 Prescribing Automorphism Groups

An automorphism of a Wilson-type STS(21) with the three STS(7) having disjoint point sets necessarily belongs to the wreath product $S_7 \wr S_3$ (of order

$3! \cdot (7!)^3$) that permutes the three point sets and the points within the sets. The basic automorphisms are as follows:

(1) an automorphism of type $1^9 2^6$ with 18 fixed blocks,
(2) an automorphism of type $1^7 2^7$ with 14 fixed blocks,
(3) an automorphism of type $1^3 2^9$ with 10 fixed blocks,
(4) an automorphism of type $1^3 3^6$ with 4 fixed blocks,
(5) an automorphism of type 3^7 with 1, 4, or 7 fixed blocks,
(6) an automorphism of type $1^7 7^2$ with 7 fixed blocks,
(7) an automorphism of type 7^3 with no fixed blocks.

Theorem 1 *A Wilson-type* STS(21) *can only possess the aforementioned basic automorphisms.*

Proof. Following [6] we observe that the fixed points of an automorphism of an STS(v) induce a subsystem of order w. When $v \neq w$, we know [9] that $v \geq 2w + 1$. Therefore, an STS(21) has automorphisms with 0, 1, 3, 7, or 9 fixed points only.

The automorphisms of a Wilson-type STS(21) should obviously be automorphisms of its STS(7) part (since there are no other STS(7) by Lemma 1). Thus, there are two possibilities: either an automorphism transforms the STS(7) into one another (automorphisms of order 2 or 3), or the STS(7) are fixed. If an STS(7) is fixed, such a transformation generates a subgroup of the automorphism group of the STS(7). The basic automorphism types of an STS(7) are $1^3 2^2$, $1^1 3^2$, and 7^1, so the only possible orders for basic automorphisms of a Wilson-type STS(21) are 2, 3, and 7.

We first consider automorphisms transforming the STS(7) X_1, X_2, X_3 into one another. If the subsystems are transformed in cycle notation as $(X_1\ X_2\ X_3)$, we obtain item 5 of the list. If the STS(7) are transformed into each other as $(X_1\ X_2)(X_3)$, there are two possibilities: If the points of the third STS(7) are fixed, we obtain item 2. If the points of the third STS(7) are transformed into each other by an automorphism of order 2, we obtain item 3.

Next we consider automorphisms that fix the three STS(7). If such an automorphism is of order 2 (and thus with 3 fixed points for each STS(7)), we get item 1. If the automorphism is of order 7, we get items 6 and 7, depending on whether this transformation acts on two or three of the STS(7). If the STS(7) automorphism is of order 3 (1 fixed point), we get item 4 with 3 fixed points in total; it is not possible for this automorphism to act only on two of the STS(7) even though $2 \cdot 9 + 1 < 21$, since that would mean that the STS(9) induced on the fixed points has a subsystem of order 7, which is clearly not possible.

In all the cases above, the number of fixed blocks follows from the necessity to provide exactly one common block for each two points. □

For each admissible automorphism, an exhaustive search was carried out to find all nonisomorphic designs. The full automorphism group of the constructed designs must be calculated, since the final design may have other automorphisms as well. For details and arguments that can be used to reduce the search space, see the partial results in [17–19].

3.3 Classification Results

The classification results following the approach in Section 3.2 are presented in Table 6 where the columns correspond to cycle types of basic automorphisms, and the number of nonisomorphic such designs with given orders of the full automorphism groups are presented. Note that designs whose full automorphism group has nonprime order may be constructed from several basic automorphisms; the count in the last column, Nd, gives the total number of nonisomorphic such designs, which in most cases differs from (is smaller than) the sum of the entries in that row.

By (2), we obtain that the number of Wilson-type STS(21) with trivial automorphism group is 2156186, so the total number of Wilson-type STS(21) is 2166351. These results were corroborated by the classification approach discussed in Section 3.1.

4 Kirkman Triple Systems of Order 21

A *resolution* of a design is a partition of the blocks into parallel classes, whose blocks in turn partition the point set. A design is said to be *resolvable* if such a resolution exists. It is called *doubly resolvable* if it has two resolutions such that each pair of parallel classes has at most one block in common. A resolution of an STS(v) is called a *Kirkman triple system of order v*, a KTS(v) for short. The Kirkman triple systems of order 21 with nontrivial automorphisms have been classified [2]. It is known that for sufficiently large $v \equiv 3 \pmod 6$, there exists a doubly resolvable KTS(v) [5], but a doubly resolvable STS(21) is still not known.

The classified Wilson-type STS(21) were checked for resolvability, and the results are compiled in Table 7. For each order of the full automorphism group of the designs, the number of resolvable such designs, Nrd, and the number

Table 6
Wilson-type STS(21) with nontrivial automorphisms

$\lvert\mathrm{Aut}\rvert$	$1^9 2^6$	$1^3 3^6$	$1^3 2^9$	3^7	$1^7 2^7$	7^3	$1^7 7^2$	Nd
2	636		7239		1039			8914
3		317		368				685
4	271		208		52			271
6		126	95	71	102			197
8	41		39		27			41
9		3		3				3
12	6	1	4	5				6
16	4		4		4			4
18		7	2	7	5			7
21		2					2	2
24	17	11	17	6	10			17
42		6			6	2	4	6
48	2	2	2		2			2
72	5	5	5	5	4			5
126		1		1	1	1		1
144	1	1	1	1	1			1
294		1			1	1	1	1
882		1		1	1	1	1	1
1008	1	1	1	1	1	1		1
Total	984	485	7617	469	1256	6	8	10165

of nonisomorphic resolutions, Nr, are given. None of the resolvable designs is doubly resolvable.

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Table 7
Resolvable Wilson-type STS(21)

|Aut|	Nrd	Nr
1	10981	12030
2	352	677
3	67	127
4	23	51
6	19	53
8	5	19
16	1	1
18	2	10
21	1	1
24	6	21
42	4	7
72	1	11
126	1	4
294	1	2
882	1	4
1008	1	18
Total	11466	13036

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