Introduction to Discrete Mathematics, Fall 2000
Problem Set 9, 21–23 November

1. Determine the capacity of the binary symmetric channel BSC(p), when (a) $p = 0.01$, (b) $p = 0.1$, (c) $p = 0.4$.

2. Show that the Hamming distance $d(x, y)$ between n-bit binary strings is a *metric* in the space $\{0,1\}^n$, i.e. for all $x, y, z \in \{0,1\}^n$ the following hold:

   (a) $d(x, y) \geq 0$,

   (b) $d(x, y) = 0$ if and only if $x = y$,

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, z) \leq d(x, y) + d(y, z)$ ("triangle inequality").

   (Of course only condition (d) is nontrivial in this case.)

3. List all the codewords of the Hamming code $\mathcal{H}_4$ (4 data bits, 3 parity bits, single-error correction), when the parity check matrix H is given in standard form. ("Standard form" = the identity matrix I_3 corresponding to the parity bits is located in the rightmost columns of the check matrix H.)

4. Prove that a single-error correcting 7-bit code can have at most 16 codewords, and hence the code $\mathcal{H}_4$ has maximal rate within this class of codes.

5. Design a single-error correcting 5-bit binary code with 4 codewords. Is the rate of this code higher or lower than that of the Hamming code $\mathcal{H}_4$?

6. Mr. K and Ms. Y are enjoying a pleasant evening in a somewhat noisy restaurant, whose noise level is modeled by the channel BSC(0.1). Mr. K would like to send Ms. Y the 4-bit message "0101".

   (a) What is the probability that the message is received correctly, if Mr. K doesn't code it in any way?

   (b) How does the situation change if Mr. K repeats the message three times, and Ms. Y chooses as the value of each bit that which occurred in the majority of repetitions? (I.e., in this case two times out of three.)

   (c) How about if Mr. K codes the message using the Hamming code $\mathcal{H}_4$? What is the coded form of the message in this case?

7. Decode the bitstring 1001010, which corresponds to a codeword of the natural $\mathcal{H}_4$ code with a one-bit error. ("Natural" = the columns of the parity check matrix H, when interpreted as binary numbers, are in numerically increasing orded.) What are the contents of the 4 "message bits" in the correct 7-bit codeword.

The second Midterm Exam of the course takes place Wed 29 November at 8–11 a.m. The exam covers the material discussed in Problem Sets 6 thru 9 (graphs and coding).