Introduction to Discrete Mathematics, Fall 2000
Problem Set 7, 7 – 9 November

1. Find some 16-bit binary De Bruijn sequence over the binary alphabet, i.e. a 16-character sequence $w \in \{0,1\}^{16}$ that, when read cyclically, contains all 4-character sequences $x \in \{0,1\}^4$ as subsequences.

2. (a) Draw pictures of all nonisomorphic 6-node trees.

   (b) Prove that there are at least e^{n-1}/n^3 distinct isomorphism types of n-node trees. (*Hint:* Apply the Sylvester–Cayley tree counting theorem and the inequality $e(\frac{n}{e})^n \leq n! \leq ne(\frac{n}{e})^n$.)

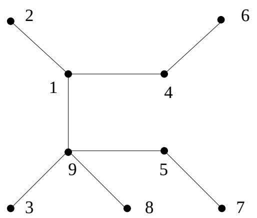
3. Prove that a graph $G = (V, E)$ is a tree, if and only if it is acyclic, but inserting any one new edge into G creates a cycle.

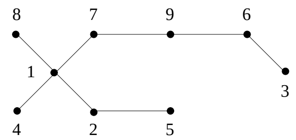
4. Prove the following claims:

   (a) Every tree with at least two nodes contains at least two leaves.

   (b) Let $G = (V, E)$ be an arbitrary graph, and $u \in V$ a leaf (= node of degree 1) in G . Then G is a tree, if and only if the graph $G - u$ (= the subgraph of G spanned by nodes $V \setminus \{u\}$) is a tree.

5. Determine the Prüfer codes of the following trees (cf. Balakrishnan, pp. 168–169):





   What is the 7-node tree corresponding to the Prüfer code (3,1,4,1,5)?

6. A *hypercube graph* H_n has as nodes all the n-bit binary sequences $u \in \{0,1\}^n$, and nodes u and v are connected by an edge if and only if the corresponding binary sequences differ in exactly one position.

   (a) Draw pictures of the graphs H_3 and H_4. How many nodes are there in graph H_n?

   (b) For which values of n does H_n contain a closed Eulerian tour?

   (c) Determine the *diameter* of graph H_n, i.e. the biggest possible internode distance in H_n, $d(H_n) = \max\{d(u,v) \mid u, v \in H_n\}$, where the *distance* $d(u,v)$ between nodes $u, v \in H_n$ is defined as the length of the shortest path connecting them.

7. Prove, by induction on n, that every hypercube graph H_n, $n \geq 2$, contains a Hamiltonian cycle. (*Hint:* Assume inductively that graph H_n contains a Hamiltonian path from node $u = 00\ldots0$ to node $v = 10\ldots0$. Observe that two such paths can be connected into a similar Hamiltonian path in H_{n+1}.) Draw pictures of the Hamiltonian paths based on this construction for the graphs H_3 and H_4.

   (*Note:* The Hamiltonian cycles of hypercube H_n correspond to so called *cyclic Gray codes* of the integers 0, ..., $2^n - 1$. These have the property that the binary representations of any two consequent numbers differ in exactly one bit.)