Introduction to Discrete Mathematics, Fall 2000
Problem Set 2, 3–5 Oct.

1. Prove the following claims:
   (a) All subsets of the set of natural numbers $\mathbf{N}$ are either finite or countably infinite.
   (b) The cartesian product $\mathbf{N} \times \mathbf{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbf{N} \times \mathbf{N}$ as laid down in a two-dimensional table. Enumerate the elements of the table along diagonals that go in the lower left to upper right direction.) *Note:* It follows easily from this result that also the set of rational numbers $\mathbf{Q}$ is countably infinite.

2. Prove that in any group of $n > 1$ people, there are at least two persons who have exactly equally many acquaintances in the group. A person is not counted among his/her own acquaintances, and we assume that the acquaintedness relation is symmetric, so that if a is acuainted with b, then also b is acquainted with b. (*Hint:* Apply the pigeonhole principle.)

3. A pack of cards contains 52 cards that are divided into four "suites" (hearts, diamonds, clubs, spades). The cards belonging to each suite are numbered 1,. . . ,13.
   (a) How many different five-card "poker hands" (five-card subsets) are there?
   (b) In how many ways is it possible to form a five-card "flush", i.e. a subset of five cards where all the cards are of the same suite?
   (c) In how many ways is it possible to form a "full house", i.e. a five-card subset consisting of a "three-of-a-kind" (three cards of the same numerical value), and a "pair" (two cards of the same numerical value)? The same card in a full house obviously cannot belong both to the pair and the three-of-a-kind combination.

4. Prove the following properties of the binomial coefficients:
   (a) $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$, for $1 \leq k \leq n$. Deduce from this fact that the binomial coefficients are increasing with respect to the index k ($\binom{n}{k-1} < \binom{n}{k}$), when $k \leq n/2$, and decreasing ($\binom{n}{k} > \binom{n}{k+1}$), when $k \geq n/2$.
   (b) Verify Pascal's formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, for $1 \leq k \leq n$, by a direct calculation based on the defining formula for the binomial coefficients.
   (c) $\sum_{i=0}^{r} \binom{n+i}{n} = \binom{n+r+1}{n+1}$, by induction on r. Which familiar formula do you obtain from this identity in the case $n = 1$? Can you come up with a combinatorial argument for proving the identity?

5. Estimate the size of the "middlemost" binomial coefficient $\binom{2n}{n}$, by approximating the factorial function with *Stirling's formula*: $n! \sim \sqrt{2\pi n}(n/e)^n$.

6. Natural numbers m ja n are *relatively prime*, if they have no common factors, i.e. if their greatest common denominator is $\gcd(m, n) = 1$. The value of *Euler's totient function* $\phi(n)$ indicates the number of natural numbers less than n that are relatively prime to n, i.e. $\phi(n) = |\{m \in \mathbf{N} : 1 \leq m \leq n, \gcd(m, n) = 1\}|$. (Thus, for instance, $\phi(5) = 4$, $\phi(6) = 2$.) Show, using the inclusion-exclusion principle, that if the different prime factors of number n are p_1, p_2,. . . , p_k, then
$$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k}).$$