A proof procedure for circumscription

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Abstract

In [Siegel and Forget, 1996] a new non monotonic logic namely X-logics was defined. In this formalism, useful for proof procedure, the non-monotonic inference relation $\vdash_X$ is defined by $A \vdash_X B$ if every classical theorem of $A \cup B$ which is in X is a theorem of A. We proved that every preferential logic is an X-logic, and we studied the particular case of circumscription without free predicate in the finite case. We found a set X smaller than the general one which simplifies the complexity of the proof procedure. In this paper, this result is extended to the case of circumscription with free predicates.

KEYWORDS: *nonmonotonic logic, preferential model approach, representation theorem, X-logic*

1 INTRODUCTION

Roughly, non-monotonic logics can be classified in two classes. First, the "preferential approach" (more semantical, like circumscription [McCarthy, 1980]), where the property of non-monotonicity comes from an order relation among models. The second approach (more syntactic, like default logics [Reiter, 1980]), where specific rules are added to describe a non-monotonic behaviour.
Non-monotonic logics can be defined by modifying the ordering relation, the form of defaults, the base language (modal, classical, numerical), etc.
Proof procedures for these different logics use generally few methods. Proving $A \mid\sim B$, comes out to find a finite subset of formulas $F = \{f_1, f_2, \ldots, f_n\}$ such that $\{A \cup F\} \vdash B$, where $\vdash$ denotes a "classical implication". This approach

close to abduction can be encountered in preferential logics, defaults logics, ATMS ... In the finite case, if the classical implication has a deduction theorem, $A \cup F \vdash B$ is equivalent to $A \wedge \neg F \vdash \neg B$. Hence proof procedures can be based either on a consequence finding algorithm [Siegel, 1987, Inoue, 1992] or on ATMS.
[Siegel, 1981, Bossu and Siegel, 1985], on the other hand, define another kind of proof procedure for sub-implication, which is a logic similar to the General Closed Word Assumption (GCWA for short). For this logic, if A is a set of clauses and B is a negative clause, $A \mid\sim B$ if and only if the set of positive clauses entailed by A and B is the same as the set of positive clauses entailed by A alone. As in the previous case, proof procedures are based on consequence finding algorithms.

In [Siegel and Forget, 1996] a generalization namely X-logics of this method is defined. In this formalism, entailment is based on classical logic. From classical entailment,it is sufficient to check wether entailed formulas are in the set X. Regarding the notion of proof procedure the idea is clearly that if it is possible to define a non monotonic logic from X-logics, it will be possible to use well known proof procedure tools of classical logics for non monotonic logics. Obviously, the complexity of the algorithm is strongly linked to the size of the set X. In this paper, we remind (1) the definition and the most important properties of X-logics, (2) the representation theorem for preferential logics, (3) the result obtained on the particular case of circumscription without free predicates. Next, we extend this result to circumscription with free predicates by proving that there exists a set X, smaller than the general one, which enable us to consider circumscription as an X-logic. We conclude and present some outlook on this work.

2 RECALLS : FORMALISM, PROPERTIES AND REPRESENTATION THEOREM

More details on this part can be found in [Siegel and Forget, 1996].

2.1 DEFINITION

In this paper, $\mathcal{L}$ is a classical logic which inference relation is denoted $\vdash$.
$\mathcal{F}$ denotes the set of all formulas of $\mathcal{L}$. If F is a set of formulas of $\mathcal{L}$, $\overline{F}$ denotes the set of theorems of F : $\overline{F} = \{f / F \vdash f\}$.
For us a classical logic is a logic which language includes the connectors $\vee$, $\wedge$, $\neg$ and $\rightarrow$, and such that:

a) it is monotonic ($\overline{A} \subset \overline{A \cup B}$),

b) it is idempotent ($\overline{\overline{A}} \subset \overline{A}$),

c) it is compact ($F \vdash f$ iff there exists a finite subset F' of F such that $F' \vdash f$),
d) it is reflexive ($A \vdash A$),
e) the deduction theorem holds:
$F \vdash f \rightarrow g$ iff $\{F \cup f\} \vdash g$.

For example, boolean algebras, first order logic, normal modal logics are classical logics.
A classical theorem of a set F of formulas is a theorem of F for $\vdash$.

Definition 1 (X-Logic) *Let X a set of formulas of $\mathcal{L}$ (X is not necessary deductively closed). The non-monotonic inference relation $\vdash_X$ is defined by :*

$$A \vdash_X B \qquad \textit{iff} \qquad \overline{A \cup B} \cap X \quad \subset \quad \overline{A} \cap X$$

Therefore $A \vdash_X B$ if every classical theorem of $A \cup B$ which is in X is a theorem of A (by adding B to A the set of classical theorems which are in X does not grow).

As the base logic is monotonic, we have $(\overline{A} \cap X) \subset (\overline{A \cup B} \cap X)$, and therefore it is possible to define $\vdash_X$ by :

$$A \vdash_X B \qquad \text{iff} \qquad \overline{A \cup B} \cap X = \overline{A} \cap X$$

Note that a monotonic logic has the following characteristic prpoperty: $\overline{A} \subset \overline{A \cup B}$. Intuitively, an X-logic keeps this property only on the set X.

Examples :

- If the set X is the set of all formulas of $\mathcal{L}$ then $\vdash_X \equiv \vdash$.
- If the set X is empty then all the formulas can be entailed.
- For the GCWA, X will be the set of positive formulas.
- In the finite propositional case, and regarding the canonical example of tweety in the case of circumscription, we consider that normal bird fly. Three propositions are used here: bird (b), abnormal (a), and fly (f). Abnormal will be the circumscribed predicate, the others being free.
  In this example,
  $X = \{a, a \wedge f, a \wedge \neg f, f \vee a, \neg f \vee a, a \wedge b, a \wedge \neg b, b \vee a, \neg b \vee a\}$.
  The detail of the construction of X is given further in the paper (cf section 3 below).We will see later why we use these formulas. Let us note A the set of known formulas (ie $A \vdash_X B$ iff $\overline{A \cup B} \cap X = \overline{A} \cap X$).
  Note the following cases:
  - If $A = \emptyset$ (We have no information).
    $A \vdash_X \neg a$ because there is no formulas entailed by $\neg a$ in X.
    It is easy to see that others formulas built on a b f entail a formula of X.

– If $A = \{b \wedge \neg a \rightarrow f\}$ in fact $A = \{\neg b \vee a \vee f\}$.
  In this case $A \vdash_X \neg a$, and $A \vdash_X \neg b \vee f$ for the same reasons.
– If $A = \{\neg b \vee a \vee f, b\}$ in fact $A = \{a \vee f, b\}$
  Here $A \vdash_X \neg a$, $A \vdash_X b$, $A \vdash_X \neg f$.
– If $A = \{\neg b \vee a \vee f, a\}$ in fact $A = \{a\}$.
  Here $A \vdash_X a$.

2.2 PROPERTIES

The following properties are all correct in the infinite case, excepted ones which need the conjunction and disjunction connectors.
Properties 1 :
For any X such that $X \neq \{True\}$ and for any classical inference$\vdash$, $\vdash_X$ has the following properties:

- Supraclassicality : If $A \vdash B$ then $A \vdash_X B$
- Reflexivity : $A \vdash_X A$
- Coherence : (If $\overline{X} \neq \{True\}$) : $True \not\vdash_X False$
- Or : If $A \vdash_X C$ and $B \vdash_X C$ then $A \vee B \vdash_X C$
- Left logical equivalent :
  If $\vdash A \leftrightarrow B$, then $(A \vdash_X C \; iff \; B \vdash_X C)$
- Right weakening :
  If $A \vdash_X B$ and $\vdash B \rightarrow C$ then $A \vdash_X C$
- Cut : If $A \vdash_X B$ and $A \wedge B \vdash_X C$ then $A \vdash_X C$
- Restricted Cumulativity :
  $A \vdash B$ iff ($A \vdash_X C$ iff $A \wedge B \vdash_X C$)

Other important properties, linked to cumulativity, are not verified.
For example the "and" property : If $A \vdash_X B$ and $A \vdash_X C$ then $A \vdash_X B \wedge C$ is not verified if A, B, C are propositional variables and $X = \{A \wedge B \wedge C\}$.
Same for Unit Cumulative Monotony (Restricted Monotonicity) : If $A \vdash_X B$ and $A \vdash_X C$ then $A \wedge B \vdash_X C$ is not verified with $X = \{A \wedge B \wedge C\}$ which is the same as previously.

The failure of these properties can be interesting, for it enables us to hope that it will be possible to capture more than cumulative logics and more than the preferential model approach (especialy default logics).

Property 2 [Siegel and Forget, 1996]:
If the complementary of X (the set of formulas of $\mathcal{L}$ which are not in X) is deductively closed, "cumulativity" and "AND" holds for $\vdash_X$.

2.3 X-LOGICS AND PREFERENTIAL MODEL APPROACH

In this section, we remember the result obtained in [Siegel and Forget, 1996] for the case or circumscription without free predicates, and we prove that it can be extended to circumscription with free predicates.

Let us first remember some definitions :

- A **preferential relation** $<$ is any relation among interpretations which is transitive (If $I < J$ and $J < K$ then $I < K$) and reflexive ($I < I$).
- An **antisymmetric preferential relation**, is a preferential relation which is antisymmetric (if $I < J$ and $J < I$ then $I = J$).
- If A is a set of formulas, a **minimal model of** A is an interpretation M which both satisfies A ($M \vdash A$) and is minimal for $<$ in the set of models of A (if M is a minimal model of A, and if M' is a model of A such that $M' < M$, then $M < M'$).
- As usual and regarding a preferential relation $<$, we define a **preferential model logic** $\vdash_<$ by $A \vdash_< B$ iff every minimal model of A satisfies B.
- If X is any set of formulas of $\mathcal{L}$, the **X-preferential relation** $<_X$ is defined such that:
  For every interpretations I and J :
  $I <_X J$
  - iff $\overline{I} \cap X \subset \overline{J} \cap X$
  - or iff $\forall f \in X, I \vdash f \Rightarrow J \vdash f$
  - or iff every formula in X true for I is true for J.

In what follows we restrict ourself to the case of a finite propositional logic.

Theorem 1 *Every preferential-model logic is an X-logic.*
(If $\vdash_<$ is a preferential logic, there exists X such that $\vdash_< \equiv \vdash_X$)
(The converse is false).

Regarding the previous representation theorem, the set X given in the proof by [Siegel and Forget, 1996] is too large to be used practically. On the other hand, in particular cases, it is possible to find a smaller set X. We proved that for the circumscription [McCarthy, 1980] of predicates $\{p_1, ..., p_n\}$, the other predicates $\{q_1, ..., q_m\}$ being fixed, X can be the set of conjunction of every clauses $P' \vee Q'$ with P' a disjunction of the p_i and Q' a disjunction of the q_j and the $\neg q_k$[Siegel and Forget, 1996].

In the next part, a new set X is given for circumscription with free predicates.

3 CIRCUMSCRIPTION WITH FREE PREDICATES

We use a semantical definition of circumscription, based on preferential models. If $\{p_1, \ldots, p_n\}$ are the circumscribed predicates, $\{q_1, \ldots, q_l\}$ are the fixed predicates and $\{r_1, \ldots, r_m\}$ are the variable predicates, $<_{circ}$ is defined by :

$I <_{circ} J$ iff a) $I/p \subset J/p$
b) $I/q = J/q$

(I/p means "the set of the circumscribed predicates true in I").
As for the r_m's, no condition is given, it is clear that $<_{circ}$ is an antisymmetric relation iff the set of variable predicates is empty.

It is known that any ordering relation can be changed to an antisymmetric ordering relation, that keeps the minimal models).
If $<$ is a preferential relation the new preferential relation $\ll$ is defined by :

a) $I \ll I$ for every I.

b) If $I < J$ and $J \nless I$ then $I \ll J$.

In the sequel $<<_{circ}$ is used to represent $<_{circ}$ (after the transformation defined above).

Theorem 2 - *Let $\mathcal{L}$ be a finite propositional language.*
- Let $\vdash_{<circ}$ be the circumscription of the predicate $\{p_1, ..., p_n\}$, the other predicates $\{r_1, ..., r_m\}$ being free.
- Let X_1 be the of formulas $(P_1 \wedge r) \vee P_2$, where P_1 is a conjunction of circumscribed predicates, P_2 is a disjunction of circumscribed predicates such that $P_1 \cup P_2 = P$ (the set of circumscribed predicates), and r is a free predicate.
- Let X_2 be the set formed by the formulas p_i.
- Let $X = X_1 \cup X_2$.
then

A) $\vdash_{<<circ} \equiv \vdash_{<X}$

B) $\vdash_{<X} \equiv \vdash_{X'}$

corollary:
In other words every circumscription logic with free predicates (all other predicates being circumscribed) is an X-logic.

Proof of the theorem
Proof of A:

We prove that $I \vdash_{<<_{circ}} J \Leftrightarrow I \vdash_{<_X} J$. We need to prove that $I <<_{circ} J \Leftrightarrow I <_X J$

1- I $<<_{\mathbf{circ}}$ J $\Leftarrow$ I $<_{\mathbf{X}}$ J

By contraposition: $I \not<<_{circ} J \Rightarrow I \not<_X J$.
We must prove that for every models I and J, there exists a formula $f \in X$ which is true in I and false in J.

Let I and J two interpretations such that $I \not<_X J$.

This means that

- Either there exists a circumscribed predicate p such that $I \vdash p$ and $J \not\vdash p$ and then $f = p$.
- or I and J are equals on p_i i.e. there exists r a free predicate such that $I \vdash r$ and $J \not\vdash r$. Then let $f = (p_1 \wedge \ldots \wedge p_k \wedge r) \vee p_{k+1} \vee \ldots \vee p_n$ where $p_1 \ldots p_k$ are the circumscribed predicates true in I, $p_{k+1} \ldots p_n$ the circumscribed predicate false in I . This formula $f \in X$, is true in I and false in J because $p_1 \ldots p_k$ are true in I and J but r is only true in I.

2- I $<<_{\mathbf{circ}}$ J $\Rightarrow$ I $<_{\mathbf{X}}$ J

Consider two interpretations I and J such that $I <<_{circ} J$ and let $f \in X$ such that $I \vdash f$. We show that $J \vdash f$.

- If f is one of the p_i, we have $I <<_{circ} J$, $I/p_i \subset J/p_i$ hence $J \vdash f$.
- If $f = (p_1 \wedge \ldots \wedge p_k \wedge r) \vee p_{k+1} \vee \ldots \vee p_n$
  - One $p_{k+1} \ldots p_n$ causes f to be true in I, but $I <<_{circ} J$, $I/p_i \subset J/p_i$ and hence $J \vdash f$.
  - all the $p_{k+1} \ldots p_n$ are false in I. But, from the hypothesis, f is true in I, hence $(p_1 \wedge \ldots \wedge p_k \wedge r)$ is true in I. By hypothesis, $I <<_{circ} J$, then all the $p_1 \ldots p_k$ are true in I and J.
    * Either one of $p_{k+1} \ldots p_n$ which is false in I is true in J, and then $J \vdash f$

* Or all the $p_{k+1} \ldots p_n$ are also false in J, hence I et J are equals on p_i.
  - Either I and J are equals on the r_m, and then $I = J$ hence $J \vdash f$.
  - Or I and J are only different on r_m and then they are not comparable which is in contradiction with the hypothesis $I <<_{circ} J$

Proof of B:

We must prove that $A \vdash_{<X} B \Leftrightarrow A \vdash_{X'} B$

1- $\mathbf{A \vdash_{<X} B \Rightarrow A \vdash_{X'} B}$

We must prove that $\overline{A \cup B} \cap X' \subset \overline{A} \cap X'$. Let $x \in \overline{A \cup B} \cap X'$. Hence x is true in every model of A and B. Let M be a model of A. We show that x is true in M.

- If M is minimal model of A, M is a model of B because $A \vdash_{<X} B$, and consequently of $A \cup B$. But x is true in every model of $A \cup B$, hence x is true in M.
- If M is not a minimal model of A, there exists I a minimal model of A such that $I < M$. From 1, $I \models x$ holds. But $x = \{\vee f_t\}$ hence there exists a f_t such that $I \models f_t$ and we showed in the first part of the proof that $M \models f_t$.

2- $\mathbf{A \not\vdash_{<X} B \Rightarrow A \not\vdash_{X'} B}$

We must prove that $\overline{A \cup B} \cap X' \not\subset \overline{A} \cap X'$, which is the same as finding a formula $x \in \overline{A \cup B} \cap X'$ such that $x \notin \overline{A}$

$A \not\vdash_{<X} B$ means that there exists M, a minimal model of A which is not a model of B.

Let $M_1 \ldots M_s$ all the minimal models of $A \cup B$. $M \not<_{X'} M_s$ because the M_s are minimal. Hence there exists $f_s \in X$ such that $M \not\models f_s$ et $M_s \models f_s$ for each s. Let $f = \vee f_s$. f is true in every M_s and false in M.

By construction $f \in X'$. f is true in every M_s hence $f \in \overline{A \cup B}$, and f is false in M, hence $f \notin \overline{A}$.

4 CONCLUSION

In this paper we showed that it is possible to find a set X smaller than the general one (see [Siegel and Forget, 1996]). In particular this X enables us to consider circumscription as an X-logic in the finite case. It should be possible to use well known tools of proof procedure for classical logic on a more general case of circumscription. It could be very interesting to study wether the set X we found is minimal, or if there exists a smaller set. The most important work that remain to be done is to find other sets X in order to represent other formalisms like defaults.

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