Hypothetical Reasoning with Defaults

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Abstract

A wide range of properties of a specification with Poole-type defaults can be formally expressed by the truth of modal formulas in its preference relation (viewed as a modal model). We show in this paper how hypothetical properties can be expressed in the same manner. These are the properties that a specification would have if axioms or defaults were added. Implicit in this result is a way of evaluating truth of such formulas: Any property of a specification with defaults is equivalent to a hypothetical property of the corresponding specification without defaults. Properties of specifications without defaults can, however, be evaluated by checking classical consequence and consistency with the axioms.

1 INTRODUCTION

Three types of consequences can be assigned to a specification containing defaults: classical, skeptical and credulous consequences. Most systems for reasoning about specifications with defaults commit the user to one or other of these types.

We present a uniform framework that is able to express and reason about a specification containing incomplete and/or inconsistent information. This framework is a conceptual tool to support the development of specifications containing defaults. It tells us, for example, whether some axioms or defaults are redundant or if a formula is (or is not) a classical, skeptical or credulous consequence of the specification or even if a certain property remains true when adding a finite number of axioms or defaults.

We assume that the specification is written in the fragment of Default Logic ([26]) using axioms and defaults of the form $true : \delta/\delta$[1]. These are Poole-type defaults without constraints (or supernormal defaults) and have been studied in [25], [6], [7], [9], [12].

The consequences of a specification with defaults are very different in nature. A logical consequence of the axioms is a more certain fact than a skeptical consequence of the specification, and this in turn is more certain than a credulous consequence. This information, however, gets lost when deciding for one consequence relation.

In order to have a coherent framework that is able to express these differences, extra machinery has to be added. For example in [15] a many-valued logic for default reasoning is presented. There the different logical values assigned to each formula reflect the degree of (un)confidence that is to be assigned to each consequence of the given default specification.

Another alternative, which we prefer, is provided by the semantics of normality models of defeasible (nonmonotonic) consequence ([28], [22], [20]). In a normality model the worlds satisfying a proposition α are related by their degree of atypicallity or abnormality. Intuitively a proposition β may be inferred from α iff β is true in every world that is minimally abnormal or atypical among those satisfying α. This has been formalized in several different ways ([10], [1], [2], [3], [4]) and its relations to standard modal logic and conditional logics investigated ([2], [21]). A general framework unifying defeasible consequence, belief revision and conditional logics has been proposed in [19]. For a discussion on the several faces of the minimality paradigm see [23].

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[1] To refer to a default $true : \delta/\delta$ it is enough to specify only the formula δ. We adopt this simplification.

For our purposes we use the fact that the process of extending information given by axioms with default information can be described semantically ([24], [27], [6]) by a preference relation between the models of the axioms. This preference relation displays the property that some such worlds are better than others since they satisfy more defaults. In this way it is as a particular "normality model". The modal formulas that are true in this modal model express relevant properties of the corresponding specification, especially if the modal language is extended with a the modal connective $\overleftarrow{\Box}$ which has the inaccessible worlds as scope. Classical, skeptical and credulous consequence and their negations can be expressed in this way.

It is a main result of this paper that any such property can also be formulated hypothetically. That is, we may ask whether a property holds for a specification or whether it would hold if a finite number of axioms or defaults (or both) were added. A method of evaluating truth of modal formulas in the preference relation associated with a specification (and therefore of determining whether the corresponding properties hold or not) is implicit in this result. In fact any property of a finite specification can be seen as a hypothetical property of the corresponding specification without defaults. Truth of the associated modal formula can thus be determined by checking classical consequence and consistency with the axioms of its subformulas.

In this work we concentrate on the case of specifications with axioms and defaults written in propositional logic.

In section 2 we present formally the concepts relevant to propositional default logic and in section 3 we illustrate how to express and reason about properties of a default specification by viewing its preference relation as a model of modal logic. We show how hypothetical properties are expressed and how they can be used to evaluate truth of modal formulas in the preference relation of a specification in section 4. A small example is presented in section 5. We conclude and refer to further work in section 6.

2 POOLE-TYPE DEFAULTS

We present formally the concepts relevant to propositional default logic. In the following L_P will denote a propositional language (or logic) built recursively by applying the connective $\Rightarrow$ to the union of the set of propositional symbols P with $\{\bot\}$ (the *falsum*). Formulas where the connectives $\wedge$, $\vee$, $\neg$ and $\Leftrightarrow$ occur are the usual abbreviations of formulas from L_P. Additionally $\top$ abbreviates $\neg\bot$.

A *valuation* is an assignment of truth values to the propositional symbols in P. Satisfaction of a formula f in a valuation V is denoted by $V \models f$. The set of valuations that satisfy a theory presentation S will be denoted by $S^\bullet$, adopting a convention from [16]. Given a set $\mathcal{V}$ of valuations of P, the set of formulas from L_P that hold in all valuations from $\mathcal{V}$ will be denoted by $\mathcal{V}^\bullet$. $S \vdash f$ denotes the fact that the formula f is a propositional consequence of the theory presentation S. The theory of a theory presentation S is the set $S^{\bullet\bullet}$ (i.e. $(S^\bullet)^\bullet$) of the propositional consequences of S (the formulas that hold in all models of S).

EXTENSIONS AND CONSEQUENCES

Specifications using axioms and defaults are pairs (A, D) where both A and D are sets of formulas of the propositional language L_P. A will be called the set of axioms and D the set of defaults. The pair (A, D) is called a *default theory presentation*.

Given a default theory presentation (A, D) the information given in the axioms is extended with the information given in the defaults in two steps. First the axioms in A are maximally extended with defaults from D, preserving consistency. Each such maximal set of axioms and defaults will be called an *extension presentation*.

Second, each extension presentation is extended with its (propositional) consequences. For each extension presentation the resulting theory is called an *extension*.

The formulas holding in *all* extensions of (A, D) are called the skeptical consequences of (A, D). Those holding in *some* extension are called the credulous consequences of (A, D). We use the notation $(A, D) \vdash_{skep} f$ (resp. $(A, D) \vdash_{cred} f$) to denote the fact that f is a skeptical consequence of (A, D) (resp. credulous).

SEMANTICS

The previous concepts can also be characterized by semantical constructions. Each default theory presentation (A, D) induces a preference relation on the models of A (the valuations where all axioms in A hold). One model is considered to be better than another iff is satisfies more defaults than the other[2]. More formally:

Definition 2.1 Given a default theory presentation (A, D), the *preference relation* $\sqsubseteq_{(A,D)} \subseteq A^\bullet \times A^\bullet$ induced by D on the models of A is the pre-order $\{(V, V') | \ V, V' \in A^\bullet$ and for all $\delta \in D$: if $V \models \delta$ then $V' \models \delta\}$. ■

The extensions of (A, D) are the formulas holding in the maximal equivalence classes of the preference re-

[2] In this way the best models are maximal models instead of minimal.

lation $\sqsubseteq_{(A,D)}$. Therefore $(A, D) \vdash_{skep} f$ iff f holds in every maximal equivalence class of $\sqsubseteq_{(A,D)}$ and $(A, D) \vdash_{cred} f$ iff f holds in some maximal equivalence class of $\sqsubseteq_{(A,D)}$ (see [11]).

3 REASONING

In this section we view the preference relation associated with a default theory presentation as a modal model. Truth of modal formulas in the preference relation of (A, D) express important properties of (A, D).

We proceed with the definition of the modal logic used (we follow [17]). The semantics is standard (Kripke), but we note that the modal logic includes a the less usual modal connective $\overleftarrow{\Box}$, the complement of $\Box$, with scope in the worlds that cannot be seen from the present one. In the following $L_P^{\overleftarrow{\Box}}$ will denote a modal propositional language (or logic) built recursively by applying the connectives $\Rightarrow$, $\Box$ and $\overleftarrow{\Box}$ to the union of the set of propositional symbols P with $\{\bot\}$. Formulas using the connectives $\wedge$, $\vee$, $\neg$, $\Leftrightarrow$ and $\Diamond$ are the usual abbreviations of formulas from $L_P^{\overleftarrow{\Box}}$ and $\top$ abbreviates $\neg\bot$. For α in $L_P^{\overleftarrow{\Box}}$ the formula $\overleftrightarrow{\Box}\alpha$ is an abbreviation of $\overleftarrow{\Box}\alpha \wedge \Box\alpha$. Two more connectives are introduced: $\overleftarrow{\Diamond}\alpha$ abbreviates $\neg\overleftarrow{\Box}\neg\alpha$ and $\overleftrightarrow{\Diamond}\alpha$ abbreviates $\neg\overleftrightarrow{\Box}\neg\alpha$.

Given a set P of propositional symbols a P-model is a triple $\mathcal{M} = (W, \sqsubseteq, M)$ where W is a non-empty set (of worlds), $\sqsubseteq$ is a binary relation on W, the visibility relation on W and M assigns to each propositional symbol in P a set of worlds from W (intuitively understood as the worlds where that propositional symbol is true).

Truth of a formula α in a world V from a P-model $\mathcal{M} = (W, \sqsubseteq, M)$ will be denoted by $\mathcal{M} \models_V \alpha$ and is defined by induction on the construction of α, as usual. For the new modal connective $\overleftarrow{\Box}$ we have $\mathcal{M} \models_V \overleftarrow{\Box}\alpha$ iff $\mathcal{M} \models_{V'} \alpha$ for every V' such that $V \not\sqsubseteq V'$ (i. e. for every V' not viewed by V). Therefore $\mathcal{M} \models_V \overleftarrow{\Diamond}\alpha$ iff $\mathcal{V} \models_{V'} \alpha$ for some V' not viewed by V, $\mathcal{M} \models_V \overleftrightarrow{\Box}\alpha$ iff $\mathcal{M} \models_{V'} \alpha$ for any world V' from $\mathcal{M}$ and $\mathcal{M} \models_V \overleftrightarrow{\Diamond}\alpha$ iff $\mathcal{M} \models_{V'} \alpha$ for some world V' from $\mathcal{M}$. Finally truth of a formula α in a P-model $\mathcal{M}$, denoted by $\mathcal{M} \models \alpha$, is the truth of α in each of the worlds of $\mathcal{M}$, as usual.

3.1 PREFERENCE AS A P-MODEL

Here we illustrate truth of the modal formulas in the preference relation associated with (A, D). This preference relation induces a particular P-model that is defined as follows:

Definition 3.1 Given a default theory presentation (A, D) the *P-model induced by* (A, D), denoted by $[\![(A, D)]\!]$ is $[\![(A, D)]\!] = (A^\bullet, \sqsubseteq_{(A,D)}, M_{A^\bullet})$ where $A^\bullet$ is the set of valuations that satisfy the axioms in A, $\sqsubseteq_{(A,D)}$ is the preference relation associated with (A, D) and $M_{A^\bullet}$ assigns to each propositional symbol p from P the set of valuations in $A^\bullet$ that assign *true* to p. ■

We now illustrate the meaning of the modal connectives in this particular modal model.

Given a propositional formula f, and a valuation V of $A^\bullet$, then $\Box f$ holds at a valuation V from $[\![(A, D)]\!]$ iff f holds in all valuations preferred to V. Since preference is measured by the defaults from D the set of valuations preferred to V is the set of valuations that model $A \cup D(V)$ where $D(V)$ is the set of defaults from D that hold in V. This is the same as stating that $[\![(A, D)]\!] \models_V \Box f$ holds iff f is a propositional consequence of $A \cup D(V)$.

In a similar way $[\![(A, D)]\!] \models_V \Diamond f$ holds iff $\neg f$ is not a propositional consequence of $A \cup D(V)$. On the other hand $[\![(A, D)]\!] \models_V \overleftarrow{\Box} f$ iff f holds in all valuations not preferred to V. These are the models of A that are not models of $D(V)$. This situation can be expressed by a formula in the case that $D(V)$ is finite. Let $\hat{D}(V)$ denote the conjunction of the formulas in $D(V)$. Then $[\![(A, D)]\!] \models_V \overleftarrow{\Box} f$ iff f is a propositional consequence of $A \cup \neg\hat{D}(V)$.

Similarly $[\![(A, D)]\!] \models_V \overleftarrow{\Diamond} f$ iff $\neg f$ is not a propositional consequence of $A \cup \neg\hat{D}(V)$.

Finally $[\![(A, D)]\!] \models_V \overleftrightarrow{\Box} f$ holds iff f is a propositional consequence of A and $[\![(A, D)]\!] \models_V \overleftrightarrow{\Diamond} f$ iff $\neg f$ is not a propositional consequence of A. This implies trivially the following.

Lemma 3.2 *Given a default theory presentation (A, D) and a propositional formula a then*

- $A \vdash a$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Box} a$ *and*
- $A \not\vdash a$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Diamond} \neg a$.

Proof Straightforward from the definition of $\overleftrightarrow{\Box}$ and $\overleftrightarrow{\Diamond}$. ■

We now characterize the property of a formula being a "implicit" default from (A, D). Whenever a default d from D holds in some world it will hold in all worlds preferred to that one. Therefore for any default from D, $[\![(A, D)]\!] \models d \Rightarrow \Box d$ holds.

However, not all formulas that satisfy this condition are defaults of D. But they are implicit defaults in the sense that they can be added to the set of defaults D without changing the induced preference relation.

Lemma 3.3 *Given a default theory presentation (A, D) and a propositional formula d then $[\![(A, D)]\!] \models d \Rightarrow \Box d$ iff $[\![(A, D)]\!] = [\![(A, D \cup \{d\})]\!]$.*

Proof (Sketch) If $[\![(A, D)]\!] \models d \Rightarrow \Box d$ then d holds in all valuations preferred to some V where d holds. This trivially implies that the preference of $(A, D \cup \{d\}$ coincides with that of (A, D).

On the other direction we have $[\![(A, D)]\!] = [\![(A, D \cup \{d\})]\!] \models d \Rightarrow \Box d$ since this is true for any default: if $V \models d$ then any V' preferred to V must satisfy d. ∎

The more important properties of credulous and skeptical consequence of a default theory presentation (A, D), and their negations follow.

Theorem 3.4 *Given a default theory presentation (A, D) and a propositional formula f then*

- $(A, D) \vdash_{cred} f$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Diamond}\Box f$;
- $(A, D) \not\vdash_{cred} f$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Box}\Diamond\neg f$;
- $(A, D) \vdash_{skep} f$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Box}\Diamond\Box f$; *and*
- $(A, D) \not\vdash_{skep} f$ *iff* $[\![(A, D)]\!] \models \overleftrightarrow{\Diamond}\Box\Diamond\neg f$.

Proof (Sketch) The proofs follow from the two important facts:

- A set of propositional formulas is an extension of (A, D) iff it is the set of propositional formulas holding in a maximal equivalence class of $\sqsubseteq_{(A,D)}$
- Above each valuation V from $\sqsubseteq_{(A,D)}$ there is V' such that the equivalence class of V' is maximal.

These facts are shown in [11] for any compact logic. We show $(A, D) \vdash_{skep} f$ iff $[\![(A, D)]\!] \models \overleftrightarrow{\Box}\Diamond\Box f$. Assume that $(A, D) \vdash_{skep} f$ i.e. assume that f holds in all extensions of (A, D). Therefore $\Box f$ holds in V' for those V' whose equivalence class is maximal. Since above any V from $[\![(A, D)]\!]$ there is such a V' we have $V \models \Diamond\Box f$. This shows that $[\![(A, D)]\!] \models \overleftrightarrow{\Box}\Diamond\Box f$.

Assume now that $[\![(A, D)]\!] \models \overleftrightarrow{\Box}\Diamond\Box f$ and let $\mathcal{E}$ be an extension of (A, D). We consider only the non trivial case of A (and, therefore $\mathcal{E}$) consistent. Take a V' that models the consistent $\mathcal{E}$. Then the equivalence class of V' is maximal and $V' \models \Diamond\Box f$. This implies that there is V'' preferred to V'' with $V'' \models \Box f$. Since the equivalence class of V' is maximal this V'' must be equivalent to V'. In this way all valuations in the maximal equivalence class of V' satisfy f (since $V'' \models \Box f$). This implies $f \in \mathcal{E}$. We have seen that f is in an arbitrary extension $\mathcal{E}$ of (A, D), i.e. we have seen that $(A, D) \vdash_{skep} f$. ∎

We can of course also express any boolean combination of the properties presented up to now. For example it is straightforward to express that a propositional formula is a credulous but not a skeptical consequence of (A, D).

We conclude the illustration of the expressive power of this framework with a final example.

In [15] a closure operator is presented that, given a default theory presentation (A, D), assigns to formulas from L_P one of six different truth values. These are related by a bilattice and formalize different intermediate levels of (un)certainty of the consequences of (A, D).

The truth values are t, for true, f for false, dt for true by default, df for false by default, $dt + df$ for true and false by default, and u for undefined.

It is straightforward to check that to a formula a truth value is assigned iff a corresponding modal formula holds in $[\![(A, D)]\!]$. For example the truth value $dt + df$ is assigned to a formula f iff f is true in some of the extensions of (A, D) and false in others. That is iff $[\![(A, D)]\!] \models (\overleftrightarrow{\Diamond}\Box f) \wedge (\overleftrightarrow{\Diamond}\Box\neg f)$.

4 HYPOTHETICAL REASONING

In this section we show how to formulate hypothetical reasoning. By this we mean the possibility of expressing in the preference relation associated with (A, D) properties of the default theories obtained by adding a finite number of axioms or defaults (or both) to (A, D).

Since any finite default theory presentation (A, D) is obtained from $(A, \emptyset)$ by adding the defaults D an important consequence follows. Properties of (A, D) can be equivalently translated to properties of $(A, \emptyset)$.

Since $(A, \emptyset)$ has no defaults this means that the meta statements about the original default theory presentation can be translated to statements about the propositional theory A.

This process can be taken a step further by incorporating in the translation the effect of the axioms in A. In this way properties of (A, D) can be equivalently translated to properties of $(\emptyset, \emptyset)$. Truth of modal formulas in the preference relation associated with $(\emptyset, \emptyset)$ corresponds to validity in the propositional version of Carnap's modal logic ([8]), since necessity is understood as truth in all possible worlds.

We firstly motivate the case of adding an axiom a to (A, D). Since $[\![(A \cup \{a\}, D)]\!]$ is the restriction of $[\![(A, D)]\!]$ to the models of a the solution is very similar to the propositional case. However, care has to be taken with the connectives $\Box$ and $\overleftarrow{\Box}$.

To achieve equivalence the scope of either $\Box$ or $\overleftarrow{\Box}$ has to be limited in such a way that only the worlds where a is true are relevant for evaluating $\Box\phi$ or $\overleftarrow{\Box}\phi$. This motivates the following definition[3].

Definition 4.1 Let a be a propositional formula. The *relativization of a modal formula ϕ to the axiom a*, denoted by $R_a^{ax}(\phi)$ is defined in the structure of the modal formulas by

- $R_a^{ax}(\bot) = \bot$,
- $R_a^{ax}(p) = p$, for $p \in P$,
- $R_a^{ax}(\psi_1 \Rightarrow \psi_2) = R_a^{ax}(\psi_1) \Rightarrow R_a^{ax}(\psi_2)$,
- $R_a^{ax}(\Box\psi) = \Box(a \Rightarrow R_a^{ax}(\psi))$,
- $R_a^{ax}(\overleftarrow{\Box}\psi) = \overleftarrow{\Box}(a \Rightarrow R_a^{ax}(\psi))$. ■

(The relativizations for the other connectives defined by abbreviations are trivially derived).

Similar considerations can be made for the case of adding a default d to (A, D). In this case again care has to be taken with the connectives $\Box$ and $\overleftarrow{\Box}$. If the present valuation V does not satisfy d no correction is needed for both $\Box$ or $\overleftarrow{\Box}$ since the valuations that are preferred to V according to $(A, D \cup \{d\})$ are precisely the same that are preferred to V according to (A, D). However, if the present valuation V satisfies d only the V' that are preferred to V according to (A, D) and also satisfy d remain preferred to V according to $(A, D \cup \{d\})$.

In this case the scope of $\Box$ has to be limited to the valuations that also satisfy d. Equivalently, the V' that are not preferred to V according to $(A, D \cup \{d\})$ are the ones that were already not preferred to V according to (A, D) or that, although being preferred to V according to (A, D) do not satisfy d.

In this way the scope of $\overleftarrow{\Box}$ has to be enlarged. It is the scope of $\overleftarrow{\Box}$ in the preference relation associated with (A, D) plus the valuations that do not satisfy d and were in the scope of $\Box$.

This motivates the following definition, where the relativizations of $\Box$ and $\overleftarrow{\Box}$ have been separated in the two cases discussed before, depending on whether the arbitrary present valuation satisfies the new default d or not.

[3]This situation has formal similarities with the reduction of many sorted logic to one sorted logic ([14]). In [18] a similar definition also occurs.

Definition 4.2 Let d be a propositional formula. The *relativization of a modal formula ϕ to the default d*, denoted by $R_d^{def}(\phi)$ is defined in the structure of the modal formulas by

- $R_d^{def}(\bot) = \bot$,
- $R_d^{def}(p) = p$, for $p \in P$,
- $R_d^{def}(\psi_1 \Rightarrow \psi_2) = R_d^{def}(\psi_1) \Rightarrow R_d^{def}(\psi_2)$,
- $R_d^{def}(\Box\psi) = \neg d \wedge \Box R_d^{def}(\psi) \vee d \wedge \Box(d \Rightarrow R_d^{def}(\psi))$,
- $R_d^{def}(\overleftarrow{\Box}\psi) = \neg d \wedge \overleftarrow{\Box} R_d^{def}(\psi) \vee d \wedge (\overleftarrow{\Box} R_d^{def}(\psi) \wedge \Box(\neg d \Rightarrow R_d^{def}(\psi)))$. ■

The previous results can be extended in a straightforward way to the case of adding a finite number of axioms and defaults. The relativization of a formula ϕ to both a set of axioms A' and a set of defaults D' is obtained by composing sequentially the relativizations to the individual axioms and defaults. The resulting relativization is denoted by $R_{(A',D')}(\phi)$.

Any property of $(A \cup A', D \cup D')$ that is expressed by truth of a modal formula in the corresponding preference relation can also be expressed by truth of a modal formula in the preference relation associated with (A, D).

Theorem 4.3 *Let A' and D' be a finite sets of propositional formulas. Then ϕ is true in the preference relation associated with $(A \cup A', D \cup D')$ iff $R_{(A',D')}(\phi)$ is true in the preference relation associated with (A, D).*

Proof The relativation to the axioms is trivial. We show only the simpler case of the relativation to one default, i.e. we show that, given a propositional formula d, for any modal formula ϕ,

$$[\![(A, D \cup \{d\})]\!] \models \phi \text{ iff } [\![(A, D)]\!] \models R_{(\emptyset,\{d\})}(\phi).$$

Note that $R_{(\emptyset,\{d\})}(\phi)$ is $R_d^{def}(\phi)$ (from definition 4.2 above). The generalization to a finite number of defaults is straightforward.

Firstly we note that the valuations in $[\![(A, D \cup \{d\})]\!]$ are precisely the ones in $[\![(A, D)]\!]$ since both theories have the same axioms.

We prove by induction on the structure of ϕ that $[\![(A, D)]\!] \models_V \phi$ iff $[\![(A, D \cup \{d\})]\!] \models_V R_d^{def}(\phi)$ for any valuation V in $[\![(A, D)]\!]$ (or $[\![(A, D \cup \{d\})]\!]$). Only the cases $\phi = \Box\psi$ and $\phi = \overleftarrow{\Box}\psi$ are interesting. Two subcases have to be considered. The first is when $V \models d$. It is straightforward to check that $V \sqsubseteq_{(A,D\cup\{d\})} V'$ iff $V \sqsubseteq_{(A,D)} V'$ and $V' \models d$. Therefore in $[\![(A, D)]\!]$ one has to restrict the scope of $\Box$ to the valuations V'

that satisfy d and are preferred to V and the scope of $\overleftarrow{\Box}$ has to be augmented with the valuations V' that are preferred to V but do not satisfy d. Therefore $[\![(A, D \cup \{d\})]\!] \models_V \Box\psi$ iff $[\![(A, D)]\!] \models_V \Box(d \Rightarrow R_d^{def}(\psi))$ and $[\![(A, D \cup \{d\})]\!] \models_V \overleftarrow{\Box}\psi$ iff $[\![(A, D)]\!] \models_V \overleftarrow{\Box}R_d^{def}(\psi) \wedge \Box(\neg d \Rightarrow R_d^{def}(\psi))$. The case $V \not\models d$ is simpler since $V \sqsubseteq_{(A,D\cup\{d\})} V'$ iff $V \sqsubseteq_{(A,D)} V'$. No correction is needed for this case and the equivalences $[\![(A, D \cup \{d\})]\!] \models_V \Box\psi$ iff $[\![(A, D)]\!] \models_V \Box R_d^{def}(\psi)$ and $[\![(A, D\cup\{d\})]\!] \models_V \overleftarrow{\Box}\psi$ iff $[\![(A, D)]\!] \models_V \overleftarrow{\Box}R_d^{def}(\psi)$ hold. The two cases are trivially generalized to the intended equivalences. ■

This result can be used in another direction. It is obvious that we can see any finite default theory presentation (A, D) as being obtained from $(\emptyset, \emptyset)$ by adding the conjunction of the axioms in A, thus obtaining $(A, \emptyset)$ and then adding the defaults in D.

In this way we may translate truth of a modal formula ϕ in the preference relation associated with (A, D) to truth of a corresponding formula in the preference relation associated with $(A, \emptyset)$ or $(\emptyset, \emptyset)$.

Corollary 4.4 *Let (A, D) be a finite default theory presentation and ϕ a formula from $L^{\overleftarrow{\Box}}$. Then*

$$[\![(A, D)]\!] \models \phi \text{ iff } [\![(A, \emptyset)]\!] \models R_{(\emptyset,D)}(\phi)$$

and

$$[\![(A, D)]\!] \models \phi \text{ iff } [\![(\emptyset, \emptyset)]\!] \models R_{(A,D)}(\phi).$$

Note that in either $[\![(A, \emptyset)]\!]$ and $[\![(\emptyset, \emptyset)]\!]$ all models are equivalent since the set of defaults is empty. In this way any subformula of the form $\overleftarrow{\Box}\phi$ evaluates to true (and $\overleftarrow{\Diamond}\phi$ to false, $\overleftrightarrow{\Box}\phi$ to $\Box\phi$ and $\overleftrightarrow{\Diamond}\phi$ to $\Diamond\phi$). Therefore modal formulas expressing properties of (A, D) can equivalently be rewritten as modal formulas without the new connective $\overleftarrow{\Box}$.

Truth of the latter is evaluated in the preference relation associated with $(A, \emptyset)$ or alternatively with $(\emptyset, \emptyset)$. Since the valuations in $[\![(A, \emptyset)]\!]$ are those that satisfy the axioms in A and are all equivalent, truth of modal formulas in this modal model relates to consequence and consistency with the axioms in A.

For a propositional formula f, $[\![(A, \emptyset)]\!] \models \Box f$ holds iff f is a propositional consequence of A. Truth of $\Diamond f$ expresses that f is not inconsistent with A. Truth of modal formulas in $[\![(\emptyset, \emptyset)]\!]$ is a special case of the above and is related to validity and satisfiability of its propositional subformulas.

It is interesting to remark that truth of modal formulas in $[\![(\emptyset, \emptyset)]\!]$ corresponds to validity in the propositional version of Carnap's modal logic. Carnap's logic, presented in [8], differs from the other modal logic formalisms from Lewis in that necessity is understood as truth in all possible worlds.

A formal account of the relation between truth of modal formulas in $[\![(A, \emptyset)]\!]$ to consequence and consistency with the axioms in A, inspired in Carnap's modal logic as presented in [18] is presented in the appendix.

5 A SMALL EXAMPLE

Here we illustrate the constructions presented previously by means of a simple example. This is a variation of the well known Nixon Diamond example. Our specification is concerned with the flying abilities of mammals, in particular bats and humans. Bats are known to fly by default whereas humans are known not to fly by default.

A particularly interesting individual is Batman (denoted by Bm) which is know to be both a bat and a human. This is modeled by the axiom $\mathit{Human}(\mathrm{Bm}) \wedge \mathit{Bat}(\mathrm{Bm})$. The relevant defaults for Batman are $\mathit{Bat}(\mathrm{Bm}) \Rightarrow \mathit{Flies}(\mathrm{Bm})$ and $\mathit{Human}(\mathrm{Bm}) \Rightarrow \neg\mathit{Flies}(\mathrm{Bm})$. Let BATMAN be the previously defined default theory presentation. It is easy to see that we cannot add consistently both defaults from BATMAN to the axiom $\mathit{Human}(\mathrm{Bm}) \wedge \mathit{Bat}(\mathrm{Bm})$. In this way two extensions are obtained, one where Bm flies and another where he doesn't. This implies that $\mathit{Flies}(\mathrm{Bm})$ is a credulous consequence, but not a skeptical one, of the specification BATMAN.

We firstly check these properties by examining truth of the corresponding modal formulas in the preference relation associated with BATMAN.

After that we relativize those formulas to the defaults $\mathit{Bat}(\mathrm{Bm}) \Rightarrow \mathit{Flies}(\mathrm{Bm})$ and $\mathit{Human}(\mathrm{Bm}) \Rightarrow \neg\mathit{Flies}(\mathrm{Bm})$.

Truth of the relativized formulas in the preference relation associated with the specification consisting only of the axiom $\mathit{Human}(\mathrm{Bm}) \wedge \mathit{Bat}(\mathrm{Bm})$ is equivalent to truth of the original formulas in the preference relation associated with BATMAN. Therefore we are provided with an alternative method of checking credulous and skeptical consequence that uses consequence and consistency with the axiom $\mathit{Human}(\mathrm{Bm}) \wedge \mathit{Bat}(\mathrm{Bm})$.

SEMANTICS

We now describe the preference relation associated with BATMAN. In this preference relation only the valuations that satisfy the axiom $\mathit{Human}(\mathrm{Bm}) \wedge \mathit{Bat}(\mathrm{Bm})$ participate.

Among those the ones that furthermore satisfy the proposition *Flies*(Bm) are equivalent since they satisfy precisely the same default $Bat(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm})$. In the same way, because of the other default $Human(\mathrm{Bm}) \Rightarrow \neg Flies(\mathrm{Bm})$ the valuations that satisfy the proposition $\neg Flies(\mathrm{Bm})$ are equivalent. There is no relationship of preference between the previous two equivalence classes and no other equivalence classes.

That *Flies*(Bm) is a credulous consequence of the BATMAN can be stated alternatively by noting that the formula $\overleftrightarrow{\Diamond}\Box Flies(\mathrm{Bm})$ holds in the associated preference relation. In fact we only have to find a valuation where $\Box Flies(\mathrm{Bm})$ holds.

Certainly in any valuation that satisfies *Flies*(Bm) this is true. We now see that the formula $\overleftrightarrow{\Box}\Diamond\Box Flies(\mathrm{Bm})$ does not hold in this preference relation and, for that reason, *Flies*(Bm) is not a skeptical consequence of the specification. It is sufficient to display a valuation where $\Diamond\Box Flies(\mathrm{Bm})$ does not hold.

Any valuation that satisfies $\neg Flies(\mathrm{Bm})$ certainly cannot satisfy $\Diamond\Box Flies(\mathrm{Bm})$.

THE RELATIVIZATIONS

We now see how to evaluate the same properties by expressing them hypothetically.

The modal formulas expressing skeptical and credulous consequence are relativized to the two defaults of BATMAN.

Here we present the general case of the relativization of the modal formulas expressing credulous and skeptical consequence to two defaults d_1 and d_2. The relativization of a formula $\overleftrightarrow{\Diamond}\Box f$ to two defaults d_1 and d_2 yields the modal formula:

$$\overleftrightarrow{\Diamond}\big(\neg d_1 \wedge \neg d_2 \wedge \Box f \ \vee\ \neg d_1 \wedge d_2 \wedge \Box(d_2 \Rightarrow f) \ \vee \\ d_1 \wedge \neg d_2 \wedge \Box(d_1 \Rightarrow f) \ \vee\ d_1 \wedge d_2 \wedge \Box(d_1 \wedge d_2 \Rightarrow f)\big).$$

The relativization of a formula $\overleftrightarrow{\Box}\Diamond\Box f$ to two defaults d_1 and d_2 yields the modal formula:

$$\begin{aligned}\overleftrightarrow{\Box}\big(&\neg d_1 \wedge \neg d_2 \wedge \Diamond(\neg d_1 \wedge \neg d_2 \wedge \Box f) \ \vee \\ &\neg d_1 \wedge \Diamond(\neg d_1 \wedge d_2 \wedge \Box(d_2 \Rightarrow f)) \ \vee \\ &\neg d_2 \wedge \Diamond(d_1 \wedge \neg d_2 \wedge \Box(d_1 \Rightarrow f)) \ \vee \\ &\quad \Diamond(d_1 \wedge d_2 \wedge \Box(d_1 \wedge d_2 \Rightarrow f))\big).\end{aligned}$$

TRUTH OF THE RELATIVIZATIONS

The relativization of the formulas $\overleftrightarrow{\Diamond}\Box Flies(\mathrm{Bm})$ and $\overleftrightarrow{\Box}\Diamond\Box Flies(\mathrm{Bm})$ to the two defaults $Bat(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm})$ and $Human(\mathrm{Bm}) \Rightarrow \neg Flies(\mathrm{Bm})$ are trivially obtained as instantiations of the previous formulas.

Their truth is to be evaluated in the preference relation associated with the specification consisting only of the axiom $Human(\mathrm{Bm}) \wedge Bat(\mathrm{Bm})$ and no defaults.

This preference relation consists of one equivalence class of all valuations that satisfy the axiom $Human(\mathrm{Bm}) \wedge Bat(\mathrm{Bm})$. In this case a number of simplifications can be made. We observe only that the operator $\overleftrightarrow{\Box}$ can be substituted by $\Box$ and $\overleftrightarrow{\Diamond}$ by $\Diamond$.

The relativization of $\overleftrightarrow{\Diamond}\Box Flies(\mathrm{Bm})$ becomes equivalent to:

$$\Diamond\big(Flies(\mathrm{Bm}) \wedge \Box(Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))\vee \\ \neg Flies(\mathrm{Bm}) \wedge \Box(\neg Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))\big)$$

and the relativization of $\overleftrightarrow{\Box}\Diamond\Box Flies(\mathrm{Bm})$ to:

$$\begin{aligned}&\Box\Big(Flies(\mathrm{Bm})\wedge \\ &\Diamond\big(Flies(\mathrm{Bm}) \wedge \Box(Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))\big) \\ &\vee \\ &\neg Flies(\mathrm{Bm})\wedge \\ &\Diamond\big(\neg Flies(\mathrm{Bm}) \wedge \Box(\neg Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))\big)\Big).\end{aligned}$$

We now proceed to evaluate truth of the modal formulas above by analyzing its proper modal subformulas (see the appendix for a formal justification of this method).

For propositional f the modal formula $\Box f$ states that f is a propositional consequence of the axioms and the formula $\Diamond f$ that f is not inconsistent with the axioms. For this reason the formula $\Box(Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))$ is true and $\Box(\neg Flies(\mathrm{Bm}) \Rightarrow Flies(\mathrm{Bm}))$ is false.

In this way the relativization of $\overleftrightarrow{\Diamond}\Box Flies(\mathrm{Bm})$ becomes equivalent to the formula $\Diamond(Flies(\mathrm{Bm}))$. This is true since *Flies*(Bm) is not inconsistent with $Human(\mathrm{Bm}) \wedge Bat(\mathrm{Bm})$. Therefore we conclude again that *Flies*(Bm) is a credulous consequence of BATMAN.

The relativization of $\overleftrightarrow{\Box}\Diamond\Box Flies(\mathrm{Bm})$ becomes equivalent to the modal formula $\Box(Flies(\mathrm{Bm}) \wedge \Diamond(Flies(\mathrm{Bm})))$ and finally to $\Box(Flies(\mathrm{Bm}))$. This is not true since *Flies*(Bm) is not a propositional consequence of the axiom $Human(\mathrm{Bm}) \wedge Bat(\mathrm{Bm})$. We conclude again that *Flies*(Bm) is not a skeptical consequence of the specification BATMAN.

6 CONCLUSIONS

We have presented a framework that is able to express a number of relevant properties of a default specification, in particular the different types of consequences (classical, skeptical and credulous).

Furthermore the framework is also able to express and answer whether such a property would hold if further axioms or defaults (or both) were added to the present specification.

This fact can be used in another direction, too. By translating any property of the specification to such a hypothetical property of the corresponding specification without defaults a means of evaluating the original properties is provided. The method checks, for the translated formula, classical consistency and consequence of the axiom part of the specification.

In this way we are provided with a flexible and uniform conceptual tool to support the development of specifications containing defaults. It tells us, for example, whether some axioms or defaults are redundant or whether a formula is a classical, skeptical or credulous consequence of the specification or even whether a certain property remains true when a finite number of axioms or defaults is added.

The implementation of the method above, on top of a classical theorem prover, is straightforward. It is enough to rewrite the original property according to the relativizations and, to determin the truth of the resulting modal formula. This directly relates to consequence and consistency with the original axioms (as stated formally in the appendix).

Two other main directions of further research should be mentioned. First the generalization of this framework to Poole-type defaults from an arbitrary compact logic is expected to be straightforward.

In fact we have used only basic properties of the preference semantics and of the satisfaction relation, and the proofs should carry without further dificulty to more general cases.

Secondly we want to generalize the class of defaults used in the specification.

The fact that Default Logic is provided with a semantics (see [13, 7]) that compares classes of models of the axioms[4] suggests that such a structure might be used in a way similar to that presented in this article.

Another generalization is to allow the specification to be provided with a priority relation among the defaults. This structure is used to select among the original extensions those that satisfy most of the best (of higher priority) defaults. To such a specification can also be assigned a preference relation (its lexicographic preference, see [5]). The only difficulty comes from the fact that the defaults are no longer independent and are related via the priority relation to other. For this reason the relativization to a finite number of defaults will depend in general also on all defaults that are related by priority to the given ones.

[4]But not models as in the preference relation.

Appendix

In the following we provide an alternative definition of truth of modal formulas (without the new connective $\overleftarrow{\Box}$) in $[\![(A, \emptyset)]\!]$ that makes explicit the dependency on consequence or consistency with the axioms in A.

This is inspired in the definition of satisfaction in Carnap's modal logic ([8]), as presented in [18]. It will be denoted by $\models_C^A$.

We note that $\models_C^\emptyset$ is precisely the satisfaction relation from the propositional version of Carnap's modal logic. We begin with the following auxiliary definition.

Definition 6.1 Let ϕ be a modal formula. The set of its modal subformulas, denoted by $modsub(\phi)$ is the set of the subformulas of ϕ of the form $\Box\psi$ that occur in ϕ, at least at one place *not* in the scope of the $\Box$ connective. ■

The definition of $\models_C^A$ follows.

Definition 6.2 For any formula ϕ, let ϕ^+ denote the formula resulting from ϕ by replacing each subformula $\psi \in modsub(\phi)$ at each place where it does not occur in the scope of a modal connective by $\top$ (true) if $\models_C^A \psi$ and by $\bot$ (false) otherwise. We now define $\models_C^A$ inductively as follows.

1. If ϕ is a classical propositional formula then $\models_C^A \phi$ iff $A \vdash \phi$ (ϕ is a propositional consequence of A);
2. If $\phi = \Box\psi$ then $\models_C^A f$ iff $A \vdash \psi^+$;
3. In all other cases $\models_C^A \phi$ iff $A \vdash \phi^+$.

The case $\phi = \Diamond\psi$ can easily be derived and evaluates to $\models_C^A \phi$ iff $A \not\vdash \neg\psi^+$. ■

Truth in the preference relation associated with $(A, \emptyset)$ and be alternatively evaluated by $\models_C^A$.

Lemma 6.3 *Given a modal formula ϕ, $[\![(A, \emptyset)]\!] \models \phi$ iff $\models_C^A \phi$.*

Proof The proof uses the property that truth of ϕ in $[\![(A, \emptyset)]\!]$ is equivalently determined by truth of ϕ^+. The equivalence $[\![(A, \emptyset)]\!] \models \phi$ iff $[\![(A, \emptyset)]\!] \models \phi^+$ will be proved below.

It is straightforward to prove that a similar property holds for $\models^A_C$. In fact $\models^A_C \phi^+$ iff $\models^A_C \phi$. Furthermore, since ϕ^+ is a propositional formula, $[\![(A,\emptyset)]\!] \models \phi^+$ iff $A \vdash \phi^+$ iff $\models^A_C \phi^+$.

The previous considerations imply that $[\![(A,\emptyset)]\!] \models \phi$ iff $[\![(A,\emptyset)]\!] \models \phi^+$ iff $\models^A_C \phi^+$ iff $\models^A_C \phi$.

We now proceed to prove, on the construction of ϕ, that given any valuation V from $[\![(A,\emptyset)]\!]$ then $[\![(A,\emptyset)]\!] \models_V \phi$ iff $[\![(A,\emptyset)]\!] \models_V \phi^+$. This trivially implies the equivalence between truth of ϕ and ϕ^+ in $[\![(A,\emptyset)]\!]$ as assumed above.

If ϕ is a propositional symbol then $\phi = \phi^+$ which implies trivially the base case. The case $\phi = \psi_1 \Rightarrow \psi_2$ follow trivially from $(\psi_1 \Rightarrow \psi_2)^+ = \psi_1^+ \Rightarrow \psi_2^+$. If $\phi = \Box\psi$ then $\Box\psi \in \mathit{modsub}(\Box\psi)$ that is $\Box\psi$ is itself one of its proper modal subformulas and consequently $(\Box\psi)^+$ is either true or false depending on whether $\models^A_C \Box\psi$ holds or not. Therefore $[\![(A,\emptyset)]\!] \models_V \Box\psi$ iff $\models^A_C \Box\psi$ iff $A \vdash \psi^+$.

The last proposition states that any V' in $[\![(A,\emptyset)]\!]$ is such that $[\![(A,\emptyset)]\!] \models_{V'} \psi^+$. By the induction hypothesis $[\![(A,\emptyset)]\!] \models_{V'} \psi^+$ iff $[\![(A,\emptyset)]\!] \models_{V'} \psi$.

Therefore $\models^A_C \Box\psi$ iff any V' in $[\![(A,\emptyset)]\!]$ is such that $[\![(A,\emptyset)]\!] \models_{V'} \psi$. That is ψ is true in all V' related to V in $[\![(A,\emptyset)]\!]$. This is equivalent to $[\![(A,\emptyset)]\!] \models_V \Box\psi$. ■

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