Covering t-sets with $(t+2)$-sets

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Abstract

New upper bounds for $C(v, k, t)$, the minimum number of k-subsets (blocks) out of a v-set such that each t-subset is contained in at least one block, are obtained by constructing corresponding covering designs with prescribed automorphisms. Computer search with a stochastic heuristic, tabu search, is used to find the covering designs. The new covering designs are listed and a table of upper bounds on $C(v, t+2, t)$ for $v \leq 28$ and $t \leq 8$ is given.

Key words: automorphism group, combinatorial optimization, covering design, tabu search

1 Introduction

We use the following standard notations. A t-(v, k, λ) *covering design* is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a family of k-subsets of X, called blocks, such that every t-subset of X is contained in at least λ blocks. The minimum number of blocks in a t-(v, k, λ) covering design is denoted by $C_\lambda(v, k, t)$ (if $\lambda = 1$, we use $C(v, k, t)$). Exact values of $C_\lambda(v, k, t)$ are known only for certain sets of parameters, for example, when the values of these parameters are relatively small. In other cases, lower and upper bounds on the value of this function are known. Upper bounds are obtained by explicitly constructing corresponding covering designs either by algebraic and combinatorial constructions, or, like in this paper, by a computer search. For recent surveys of covering designs, see [17,23].

We formulate the problem of constructing covering designs as a combinatorial optimization problem, where a global optimum corresponds to a desired cover-

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ing design. In trying to solve these optimization instances, we use a stochastic local search heuristic, *tabu search*. Furthermore, we prescribe automorphisms of desired covering designs to facilitate the search; then we try to find sets of orbits of blocks leading to record-breaking bounds.

The authors have recently used this method [20] to improve upper bounds on $C(v, t+1, t)$. We here continue this research and consider the function $C(v, t+2, t)$, that is, we want to cover all t-sets of a v-set using $(t+2)$-sets. A large number of new bounds is found, which lead to other improvements by different combinatorial methods. These update parts of the tables in [11].

We survey the search method in Section 2. In Section 3 we discuss some old bounds and constructions. Finally, in Section 4, we give a table of upper bounds on $C(v, t+2, t)$ for $v \leq 28$ and $t \leq 8$ and list the new covering designs.

2 Searching for covering designs

In this section, we give an overview of our search method for covering designs with prescribed automorphisms. This approach has been treated in-depth in [20], to which we refer the interested reader. See also [8].

The use of non-exhaustive heuristics in searches for covering designs was considered by Nurmela and Östergård in [19] (Stanton and Bate [22] had earlier used exhaustive methods for finding optimal covering designs). In [19] the number of points was limited to $v \leq 13$; it turned out that for larger values of v, the search did not converge to good solutions in a desired way. To limit the search space, we may require the covering design to have certain automorphisms. Analogous approaches of prescribing automorphisms before (exhaustive or non-exhaustive) searches have earlier turned out to work very well both for t-designs (in, for example, [13,14]) and packing designs (in, for example, [5,18]); also for covering designs, the results are promising [20].

Our method is based on the results by Kramer and Mesner [13]. They noted that the problem of constructing designs with prescribed automorphisms can be conveniently defined using a so-called A_{tk}-matrix. In this matrix, the rows correspond to the orbits of t-sets out of the v-set under the action of a given permutation group G. Similarly, the columns correspond to the orbits of k-sets out of the v-set under the action of G. An entry of the A_{tk}-matrix tells how many times the t-set orbit is contained in the k-set orbit. Then there is a t-design with parameters t-(v, k, λ) with (not necessarily full) automorphism group G exactly when $A_{tk}x = \lambda J$ has a solution (x is a column vector with positive, integer entries, and J is the all-one column vector). To find packing designs, we have to solve $A_{tk}x \leq \lambda J$ (see [5]), and to find covering designs [20]

$$A_{tk}x \geq \lambda J. \tag{1}$$

Given the parameters of a covering design and a group G, the problem is thus reduced to that of finding a solution to $A_{tk}x \geq \lambda J$. Furthermore, we want to minimize the size of the covering design. Hence, we introduce a row vector c containing the sizes of the k-set orbits. Then the size of the covering design is cx, which we want to minimize among the solutions of (1).

Advanced methods have been developed to attack the system of equations obtained [8]. The methods used have mainly been exhaustive. Unfortunately, most cases considered in this paper lead to so large A_{tk} matrices that exhaustive methods cannot be used. We have thus developed a heuristic search method based on *tabu search* [9].

Our search algorithm starts from a random collection of orbits of k-sets containing fewer blocks than the best previously known covering design. The algorithm finds an uncovered orbit of a t-set and then considers all possible changes of one orbit in the current solution such that the orbit of the uncovered t-set is covered. A change that leaves fewest other orbits of t-sets uncovered is selected, and the solution is modified accordingly. If we at any stage find out that there are no uncovered orbits of t-sets, we have a covering design and the search can be ended. The number of blocks is constant during the search, so in addition to the automorphism group we prescribe the sizes of the orbits in the desired covering design. A tabu list with attributes of recent moves is used to prevent returning to recently examined solutions.

3 Some old bounds

This paper is devoted to upper bounds for covering designs. However, we shall here present one central lower bound, the *Schönheim lower bound.*

Theorem 1 $C_\lambda(v,k,t) \geq L_\lambda(v,k,t) = \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \cdots \lceil \frac{v-t+1}{k-t+1}\lambda \rceil \cdots \rceil \rceil$.

We write $L(v,k,t)$ if $\lambda = 1$. The Schönheim bound is met for many small parameters. Trivially, $C(v,3,1) = \lceil v/3 \rceil$. Mills [15,16] (with many independent results by Horton, Mullin, and Stanton [12]) showed that $C(v,4,2) - L(v,4,2)$ takes the values 0, 1, and 2 for $v \notin \{7,9,10,19\}$, $v \in \{7,9,10\}$, and $v = 19$, respectively. Hence we restrict the main table to $t \geq 3$. Some results are known when $v - k$ is small. Theorems 2 and 3 are not difficult to prove; Theorem 4 was proved by Turán [24].

Theorem 2 $C(v, v, t) = 1$.

Theorem 3 $C(v, v-1, t) = t + 1$.

Theorem 4 $C(v, v-2, t) = L(v, v-2, t)$.

The following important result was proved by Schönheim [21] (note that the design mentioned in the theorem is a t-design).

Theorem 5 *If there exists a t-$(v, k, 1)$ design, then* $C(v+1, k, t) = L(v+1, k, t)$.

When there is a t-$(v, k, 1)$ design, then the best known upper bound for $C(v-1, k, t)$ in some cases comes from Theorem 6.

Theorem 6 $C(v, k, t) \leq C(v+1, k, t)$.

In the sequel, some constructions are discussed where covering designs (for example, those listed in Table 3) are used to get other covering designs. These constructions have been discussed in [11] and [20] and are left without proofs.

Theorem 7 $C(v, k, t) \leq C(v-1, k, t) + C(v-1, k-1, t-1)$.

Note that this is not a strong theorem in the following sense. The bound follows by taking the blocks of a $(t-1)$-$(v-1, k-1, 1)$ and a t-$(v-1, k, 1)$ covering design and by adding a new point to all blocks of the former one (which then cover all t-subsets containing the new point, while the rest of the blocks cover all t-subsets not containing the new point). Clearly, since the blocks containing the new point also cover t-subsets not containing the new point, instead of a t-$(v-1, k, 1)$ covering design, a partial covering will suffice in the construction. In practice, new bounds obtained using Theorem 7 can often be further improved by taking the new covering design, deleting some blocks, and applying heuristic computer search.

In the next constructions (see [11]), the distribution of points affects the result.

Theorem 8 *If there is a $(t+1)$-$(v+1, k+1, 1)$ covering design with a point occurring in M_1 blocks, then $C(v, k, t) \leq M_1$.*

As we can take $M_1 \leq \lfloor (k+1)C(v+1, k+1, t+1)/(v+1) \rfloor$ if we start from an optimal $(t+1)$-$(v+1, k+1, 1)$ covering design, we get a useful corollary.

Corollary 9 $C(v, k, t) \leq \lfloor (k+1)C(v+1, k+1, t+1)/(v+1) \rfloor$.

Sidorenko (see [11]) proposed the following construction, which is weak in the same sense as Theorem 7. It is not as useful here as for getting bounds on $C(v, t+1, t)$ (in [20]). The larger $k - t$ grows, the less useful the construction gets.

Theorem 10 *If there is a* $(t-1)$-$(v-1,k-1,1)$ *covering design with* M *blocks and a point occurring in* M_1 *blocks, then* $C(v,k,t) \le 2M - M_1 + C(v-2,k,t)$.

A corollary is obtained using the fact that there must exist an optimal $(t-1)$-$(v-1,k-1,1)$ covering design that has a point for which $M_1 \ge \lceil (k-1)C(v-1,k-1,t-1)/(v-1) \rceil$.

Corollary 11 $C(v,k,t) \le \lfloor (2v-k-1)C(v-1,k-1,t-1)/(v-1) \rfloor + C(v-2,k,t)$.

In [20] it was shown how to apply this construction repeatedly. Here such a consideration is needed only in one case, described in Example 1.

Example 1. The bound $C(21,7,5) \le 1359$ is obtained using Theorem 10. Using Corollary 11, we further get $C(22,8,6) \le 4260$. But we can do better. The covering design giving $C(20,6,4) \le 400$ in [10] has $\min_{x,y\in X, x\ne y}(3|\{B \in \mathcal{B} \mid x \notin B, y \notin B\}| + |\{B \in \mathcal{B} \mid x \notin B, y \in B\}| + |\{B \in \mathcal{B} \mid x \in B, y \notin B\}|) = 656$. Then [20, Theorem 3] gives $C(22,8,6) \le 400 + 656 + 707 + 446 + 1995 = 4204$.

To apply Theorems 8 and 10, we need to know the point distributions of the covering designs involved. Information on the covering designs used and the distributions are given in Table 1. Note that there may exist several nonisomorphic covering designs with the parameters given in the first two columns of this table, so it is essential to point out the covering design used (in the third column). In the last two columns, the maximum and minimum number of occurrences of any point in the covering design are given. If one of these values is not needed in the constructions, the entry is empty.

4 The main tables

The new covering designs found using the method described in Section 2 are listed in Table 3 with orbit representatives in hexadecimal notation. Leading 0's are discarded. The numbering of the positions goes from right (0) to left $(n-1)$. Furthermore, the orbit sizes are shown as superscript. For each covering design, generating permutations and the name of the corresponding group (following [6]) are given.

The new covering designs found and other covering designs derived from these lead to a vast number of new upper bounds for $C(v,t+2,t)$. Best known upper bounds on $C(v,t+2,t)$ for $v \le 28$, $3 \le t \le 8$ are tabulated in Table 2. A

Table 1
Point distributions of covering designs

Covering design	$\lvert\mathcal{B}\rvert$	Reference	$\max\{M_1\}$	$\min\{M_1\}$
7-$(13, 9, 1)$	79	[19]		51
7-$(15, 9, 1)$	291	[2]		161
4-$(17, 6, 1)$	188	[3]		65
7-$(18, 9, 1)$	1368	This paper		680
6-$(19, 8, 1)$	1507	[10]	641	
4-$(20, 6, 1)$	400	[3]	148	
7-$(21, 9, 1)$	5472	This paper	2364	1995
5-$(22, 7, 1)$	1727	[10]	596	
5-$(26, 7, 1)$	4353	[10]	1468	

period indicates an exact value. Only one reference is given for each bound. One of our new bounds, $C(15, 10, 8) \leq 270$, has been independently obtained by Rade Belić.

Table 2

Upper bounds for $C(v,t+2,t)$, $v \leq 28$, $3 \leq t \leq 8$

$v \backslash t$	3	4	5	6	7	8
5	$1.^b$					
6	$4.^c$	$1.^b$				
7	$5.^d$	$5.^c$	$1.^b$			
8	$8.^e$	$7.^d$	$6.^c$	$1.^b$		
9	$12.^f$	$12.^e$	$9.^d$	$7.^c$	$1.^b$	
10	$17.^f$	$20.^f$	$20.^e$	$12.^d$	$8.^c$	$1.^b$
11	$20.^m$	$32.^n$	34^m	$29.^e$	$15.^d$	$9.^c$
12	29^n	41^n	59^n	51^{m1}	$40.^e$	$18.^d$
13	34^n	66^n	78^h	101^{v1}	79^n	$52.^e$
14	43^h	80^h	138^{v1}	161^{m1}	166^{v6}	119^{v5}
15	57^m	118^h	190^{v4}	283^{v4}	280^m	270^a
16	65^{m1}	152^a	288^a	408^a	550^{v1}	448^a
17	$68.^s$	188^h	405^a	680^{m1}	884^a	998^t
18	$94.^p$	236^h	593^t	918^h	1368^a	1629^a
19	108^{v2}	330^t	707^h	1507^{v6}	2286^t	2628^a
20	133^a	400^h	1003^{v6}	1995^{m1}	3741^{k1}	4644^a
21	151^h	502^{v1}	1359^{k1}	2998^t	5472^a	8385^t
22	176^a	580^h	1727^{v6}	4204^{k2}	8470^t	13224^{k1}
23	187^a	723^{v1}	2268^{v1}	5856^{k1}	11891^a	21694^t
24	234^h	784^h	2918^{v1}	7997^u	17457^a	32353^k
25	257^{v3}	1018^t	3600^a	10350^a	23700^a	48576^a
26	$260.^s$	1170^i	4353^{v1}	13800^a	33900^a	70000^a
27	$319.^p$	$1170.^s$	4680^i	17588^{k1}	45279^a	97929^a
28	372^k	$1489.^p$	$4680.^s$	21762^a	62127^a	140049^a

. — Exact value
a — This paper
b — Theorem 2
c — Theorem 3
d — Theorem 4
e — Boyer *et al.* [4]
f — Bate and van Rees [1]
h — Bluskov and Hämäläinen [3]
i — Theorem 6
k — Corollary 11
$k1$ — Theorem 10
$k2$ — Example 1
m — Corollary 9
$m1$ — Theorem 8
n — Nurmela and Östergård [19]
p — Theorem 5
s — Steiner system [7,25]
t — Theorem 7
u — Unpublished, constructed by Rade Belić [2]
vi — Various constructors [10]: 1) Rade Belić, 2) Roy Gourgi, 3) Floyd Oats, 4) Dietmar Pree, 5) Alexander Sidorenko, 6) Rade Belić and Dietmar Pree

Table 3
New covering designs

Bound [Group] Generators	Base blocks
$C(15,10,8) \leq 270$ $[\mathbb{Z}_{15}]$ (0,1,2,3,4,5,6,7,8,9,10, 11,12,13,14)	BEF^{15}, EFB^{15}, $\mathrm{F3F}^{15}$, $\mathrm{FF5}^{15}$, $\mathrm{137F}^{15}$, $\mathrm{15EF}^{15}$, $\mathrm{16FD}^{15}$, $\mathrm{17BB}^{15}$, $\mathrm{17D7}^{15}$, $\mathrm{19FB}^{15}$, $\mathrm{1ABF}^{15}$, $\mathrm{1BD7}^{15}$, $\mathrm{1D77}^{15}$, $\mathrm{1DBD}^{15}$, $\mathrm{1DDD}^{15}$, $\mathrm{1F5B}^{15}$, $\mathrm{1F6D}^{15}$, $\mathrm{2DD7}^{15}$
$C(16,6,4) \leq 152$ $[\mathbb{Z}_{16}]$ (0,1,2,3,4,5,6,7,8,9,10, 11,12,13,14,15)	BB^{16}, $\mathrm{50F}^{16}$, $\mathrm{5A5}^{16}$, $\mathrm{671}^{16}$, $\mathrm{70D}^{16}$, $\mathrm{8D5}^{16}$, $\mathrm{A4B}^{16}$, $\mathrm{14C9}^{16}$, $\mathrm{1313}^{8}$, $\mathrm{1515}^{8}$, $\mathrm{1919}^{8}$
$C(16,7,5) \leq 288$ $[\mathbb{Z}_{16}]$ See $C(16,6,4)$	$\mathrm{16F}^{16}$, $\mathrm{375}^{16}$, $\mathrm{3E5}^{16}$, $\mathrm{69B}^{16}$, $\mathrm{9C7}^{16}$, $\mathrm{A3D}^{16}$, $\mathrm{AB9}^{16}$, $\mathrm{BA3}^{16}$, $\mathrm{E69}^{16}$, $\mathrm{113B}^{16}$, $\mathrm{12CB}^{16}$, $\mathrm{1347}^{16}$, $\mathrm{14AD}^{16}$, $\mathrm{151D}^{16}$, $\mathrm{1595}^{16}$, $\mathrm{1653}^{16}$, $\mathrm{1993}^{16}$, $\mathrm{1D49}^{16}$
$C(16,8,6) \leq 408$ $[\mathbb{Z}_{16}]$ See $C(16,6,4)$	$\mathrm{3ED}^{16}$, $\mathrm{4BF}^{16}$, $\mathrm{577}^{16}$, $\mathrm{AD7}^{16}$, $\mathrm{DE3}^{16}$, $\mathrm{E79}^{16}$, $\mathrm{11B7}^{16}$, $\mathrm{11BB}^{16}$, $\mathrm{11F9}^{16}$, $\mathrm{135B}^{16}$, $\mathrm{139D}^{16}$, $\mathrm{159B}^{16}$, $\mathrm{16CB}^{16}$, $\mathrm{1739}^{16}$, $\mathrm{1765}^{16}$, $\mathrm{193D}^{16}$, $\mathrm{19D3}^{16}$, $\mathrm{19D5}^{16}$, $\mathrm{1D4D}^{16}$, $\mathrm{1EA5}^{16}$, $\mathrm{2667}^{16}$, $\mathrm{269D}^{16}$, $\mathrm{3355}^{16}$, $\mathrm{F0F}^{8}$, $\mathrm{1717}^{8}$, $\mathrm{1B1B}^{8}$, $\mathrm{2B2B}^{8}$, $\mathrm{2D2D}^{8}$
$C(16,10,8) \leq 448$ $[\mathbb{Z}_{16}]$ See $C(16,6,4)$	$\mathrm{13DF}^{16}$, $\mathrm{15BF}^{16}$, $\mathrm{17F9}^{16}$, $\mathrm{19F7}^{16}$, $\mathrm{1B7D}^{16}$, $\mathrm{1D7B}^{16}$, $\mathrm{1DDD}^{16}$, $\mathrm{1FB5}^{16}$, $\mathrm{1FD3}^{16}$, $\mathrm{27EB}^{16}$, $\mathrm{2BBB}^{16}$, $\mathrm{2BE7}^{16}$, $\mathrm{2E7B}^{16}$, $\mathrm{2EB7}^{16}$, $\mathrm{333F}^{16}$, $\mathrm{337B}^{16}$, $\mathrm{33B7}^{16}$, $\mathrm{35DB}^{16}$, $\mathrm{36AF}^{16}$, $\mathrm{373B}^{16}$, $\mathrm{379D}^{16}$, $\mathrm{3ABB}^{16}$, $\mathrm{3BAB}^{16}$, $\mathrm{3BD5}^{16}$, $\mathrm{555F}^{16}$, $\mathrm{55D7}^{16}$, $\mathrm{1F1F}^{8}$, $\mathrm{2F2F}^{8}$, $\mathrm{3D3D}^{8}$, $\mathrm{5B5B}^{8}$
$C(17,7,5) \leq 405$ $[\mathbb{Z}_{15}]$ See $C(15,10,8)$	$\mathrm{17D}^{15}$, $\mathrm{4ED}^{15}$, $\mathrm{5B9}^{15}$, $\mathrm{61F}^{15}$, $\mathrm{AC7}^{15}$, $\mathrm{D47}^{15}$, $\mathrm{126B}^{15}$, $\mathrm{1327}^{15}$, $\mathrm{1553}^{15}$, $\mathrm{80D7}^{15}$, $\mathrm{8339}^{15}$, $\mathrm{83A3}^{15}$, $\mathrm{854B}^{15}$, $\mathrm{8789}^{15}$, $\mathrm{88AD}^{15}$, $\mathrm{8A4D}^{15}$, $\mathrm{100EB}^{15}$, $\mathrm{10273}^{15}$, $\mathrm{1028F}^{15}$, $\mathrm{1049B}^{15}$, $\mathrm{10695}^{15}$, $\mathrm{108B5}^{15}$, $\mathrm{10B23}^{15}$, $\mathrm{180D9}^{15}$, $\mathrm{18117}^{15}$, $\mathrm{18433}^{15}$, $\mathrm{18525}^{15}$
$C(17,9,7) \leq 884$ $[\mathbb{Z}_{16}]$ See $C(16,6,4)$	$\mathrm{1FF}^{16}$, $\mathrm{7BB}^{16}$, $\mathrm{7F5}^{16}$, $\mathrm{BB7}^{16}$, $\mathrm{CF7}^{16}$, $\mathrm{D3F}^{16}$, DAF^{16}, $\mathrm{F5D}^{16}$, $\mathrm{F6D}^{16}$, $\mathrm{12F7}^{16}$, $\mathrm{15D7}^{16}$, $\mathrm{16DD}^{16}$, $\mathrm{18FD}^{16}$, $\mathrm{1A9F}^{16}$, $\mathrm{1BE9}^{16}$, $\mathrm{1CEB}^{16}$, $\mathrm{1DCD}^{16}$, $\mathrm{1E67}^{16}$, $\mathrm{24DF}^{16}$, $\mathrm{265F}^{16}$, $\mathrm{2977}^{16}$, $\mathrm{29DB}^{16}$, $\mathrm{2ABD}^{16}$, $\mathrm{2ACF}^{16}$, $\mathrm{2BD5}^{16}$, $\mathrm{2D9D}^{16}$, $\mathrm{34F5}^{16}$, $\mathrm{356B}^{16}$, $\mathrm{35A7}^{16}$, $\mathrm{103ED}^{16}$, $\mathrm{104BF}^{16}$, $\mathrm{10577}^{16}$, $\mathrm{10AD7}^{16}$, $\mathrm{10DE3}^{16}$, $\mathrm{10E79}^{16}$, $\mathrm{111BB}^{16}$, $\mathrm{111DB}^{16}$, $\mathrm{111F5}^{16}$, $\mathrm{1135D}^{16}$, $\mathrm{113D5}^{16}$, $\mathrm{11579}^{16}$, $\mathrm{1159B}^{16}$, $\mathrm{116CB}^{16}$, $\mathrm{11765}^{16}$, $\mathrm{11937}^{16}$, $\mathrm{11973}^{16}$, $\mathrm{119D5}^{16}$, $\mathrm{11EA5}^{16}$, $\mathrm{12673}^{16}$, $\mathrm{1269D}^{16}$, $\mathrm{13355}^{16}$, $\mathrm{10F0F}^{8}$, $\mathrm{11717}^{8}$, $\mathrm{11B1B}^{8}$, $\mathrm{11D1D}^{8}$, $\mathrm{12727}^{8}$, $\mathrm{12B2B}^{8}$, $\mathrm{12D2D}^{8}$, $\mathrm{13535}^{8}$, $\mathrm{13333}^{4}$
$C(18,9,7) \leq 1368$ $[F_{9,8}]$ (1,3,7,8,2,6,5,4)(10,12, 16,17,11,15,14,13), (0,1, 2)(3,4,5)(6,7,8)(9,10,11) (12,13,14)(15,16,17)	$\mathrm{6EF}^{72}$, $\mathrm{E7B}^{72}$, EDD^{72}, $\mathrm{EF9}^{72}$, $\mathrm{3657}^{72}$, $\mathrm{367C}^{72}$, $\mathrm{370F}^{72}$, $\mathrm{37D4}^{72}$, $\mathrm{9E4B}^{72}$, $\mathrm{AE1D}^{72}$, $\mathrm{AE2E}^{72}$, $\mathrm{AE33}^{72}$, $\mathrm{AEE8}^{72}$, $\mathrm{BE62}^{72}$, $\mathrm{BE92}^{72}$, $\mathrm{BFC0}^{72}$, $\mathrm{FE41}^{72}$, $\mathrm{777}^{36}$, $\mathrm{37CA}^{36}$, $\mathrm{AE8B}^{36}$, $\mathrm{FE09}^{36}$

Table 3
New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(18,10,8) \leq 1629$ $[F_{9,2}]$ (1,2)(3,6)(4,8)(5,7)(10,11)(12,15)(13,17)(14,16), (0,1,2)(3,4,5)(6,7,8)(9,10,11)(12,13,14)(15,16,17), (0,3,6)(1,4,7)(2,5,8)(9,12,15)(10,13,16)(11,14,17)	E7F^{18}, EFE^{18}, 157F^{18}, 1B7E^{18}, 1BBB^{18}, 1BF5^{18}, 376D^{18}, 3773^{18}, 3ADB^{18}, 3BC7^{18}, 3C5F^{18}, 3DC7^{18}, 3E3B^{18}, 3E3D^{18}, 7E53^{18}, 7EC3^{18}, 7EC5^{18}, 92BF^{18}, 92DF^{18}, 95EB^{18}, 9CDE^{18}, 9D73^{18}, 9E79^{18}, 9E97^{18}, 9EBA^{18}, 9F3C^{18}, A63F^{18}, A6EE^{18}, A8BF^{18}, A8EF^{18}, AE5D^{18}, AEB3^{18}, AEE9^{18}, AF2E^{18}, B4F5^{18}, B56D^{18}, B6CB^{18}, B6D9^{18}, BAAD^{18}, BADC^{18}, BAE3^{18}, BB8E^{18}, BC75^{18}, BCB5^{18}, BD56^{18}, BD5C^{18}, BDA9^{18}, BDAA^{18}, BE1E^{18}, C4BF^{18}, CE7A^{18}, CE8F^{18}, CEBC^{18}, CF39^{18}, D69E^{18}, D6CD^{18}, D6F2^{18}, D793^{18}, DCAE^{18}, DCEC^{18}, EC57^{18}, ECA7^{18}, ECB9^{18}, EDCA^{18}, FE07^{18}, FE8C^{18}, 1B61D^{18}, 1B665^{18}, 1B66C^{18}, 1BA95^{18}, 1BE49^{18}, 1BEA1^{18}, 1DC1D^{18}, 1DC2D^{18}, 1DC71^{18}, 1DE54^{18}, 1DE62^{18}, 1EA1E^{18}, 1EE19^{18}, 1EE2C^{18}, 1EEC8^{18}, 1FE50^{18}, 36EE^{9}, 3BB6^{9}, 7ECA^{9}, 9E4F^{9}, A7F8^{9}, AF74^{9}, BA76^{9}, C47F^{9}, CE67^{9}, D66B^{9}, EDD4^{9}, FFC0^{9}, 1B62E^{9}, 1BF24^{9}, 1DC1B^{9}, 1EE07^{9}, 1EE92^{9}
$C(19,10,8) \leq 2628$ $[F_{9,8}]$ See $C(18,9,7)$	3C77^{72}, 3C7D^{72}, 3CB7^{72}, 3CD7^{72}, 3CDB^{72}, 3DE3^{72}, 7E4E^{72}, 7EC3^{72}, 9E57^{72}, 9E7C^{72}, 9EB9^{72}, 9F3A^{72}, FEA1^{72}, 406EF^{72}, 40E7B^{72}, 40EDD^{72}, 40EF9^{72}, 4366B^{72}, 436CD^{72}, 43759^{72}, 43766^{72}, 49E17^{72}, 4AE1D^{72}, 4AE2E^{72}, 4AE33^{72}, 4AEE8^{72}, 4BE62^{72}, 4BE92^{72}, 4FE41^{72}, 7FB^{36}, EEF^{36}, 3C3F^{36}, 7E1B^{36}, 7EE4^{36}, 9EA7^{36}, FE54^{36}, 1FE0A^{36}, 40777^{36}, 43735^{36}, 43CAD^{36}, 47E54^{36}, 4AE8B^{36}, 4FE09^{36}, 4FF80^{36}
$C(20,5,3) \leq 133$ $[\mathbb{Z}_{19}]$ (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18)	331^{19}, 459^{19}, 110D^{19}, 1C21^{19}, 2095^{19}, 80981^{19}, 80A41^{19}
$C(20,10,8) \leq 4644$ $[F_{9,8}]$ See $C(18,9,7)$	EFD^{72}, 375E^{72}, 37C7^{72}, 7E53^{72}, 7EC5^{72}, AE4F^{72}, AE79^{72}, AEAD^{72}, AEBA^{72}, BE53^{72}, BE99^{72}, BECA^{72}, BFA8^{72}, 1FF01^{72}, 406F7^{72}, 40E6F^{72}, 40EFA^{72}, 4366B^{72}, 436CD^{72}, 43759^{72}, 43766^{72}, 4AE1D^{72}, 4AE2E^{72}, 4AE33^{72}, 4AEE8^{72}, 4BF24^{72}, 4FE44^{72}, 80E7B^{72}, 80EDD^{72}, 80EEB^{72}, 8366B^{72}, 836CD^{72}, 83759^{72}, 83766^{72}, 8AE1D^{72}, 8AE2E^{72}, 8AE33^{72}, 8AEE8^{72}, 8BF24^{72}, 8FE44^{72}, C02BF^{72}, C063F^{72}, C0EF1^{72}, C1F0B^{72}, C361E^{72}, C36C5^{72}, C3746^{72}, C3758^{72}, C3E38^{72}, C3E49^{72}, C3F11^{72}, 7FB^{36}, 363F^{36}, 36F5^{36}, 9F59^{36}, AEDC^{36}, FE54^{36}, 407AF^{36}, 43735^{36}, 49EA6^{36}, 4AE8B^{36}, 4FE14^{36}, 83735^{36}, 8AE8B^{36}, 8FE14^{36}, C07F8^{36}, C0E4F^{36}, C0E67^{36}, C0EDC^{36}, C0EE9^{36}, C0EF2^{36}, C3635^{36}, C7EC0^{36}, C9EA1^{36}, C9F11^{36}, CAE07^{36}, CAE8C^{36}, CFE40^{36}

Table 3

New covering designs (continued)

Bound [Group] Generators	Base blocks
$C_1(21,9,7) \leq 5472$ $[F_{19,9}]$ (1,4,16,7,9,17,11,6,5)(2,8,13,14,18,15,3,12,10), (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18)	AFB^{171}, $\mathrm{F67}^{171}$, $\mathrm{195F}^{171}$, $\mathrm{19B7}^{171}$, $\mathrm{1AED}^{171}$, $\mathrm{1C9F}^{171}$, $\mathrm{1D2F}^{171}$, $\mathrm{1ED3}^{171}$, $\mathrm{1F39}^{171}$, $\mathrm{37C9}^{171}$, $\mathrm{805CF}^{171}$, $\mathrm{8079D}^{171}$, $\mathrm{809F9}^{171}$, $\mathrm{80BA7}^{171}$, $\mathrm{80E5D}^{171}$, $\mathrm{8127B}^{171}$, $\mathrm{812AF}^{171}$, $\mathrm{1002DF}^{171}$, $\mathrm{1004F7}^{171}$, $\mathrm{1006BD}^{171}$, $\mathrm{10075B}^{171}$, $\mathrm{1007E9}^{171}$, $\mathrm{1009DD}^{171}$, $\mathrm{100F99}^{171}$, $\mathrm{101957}^{171}$, $\mathrm{101B95}^{171}$, $\mathrm{1801BB}^{171}$, $\mathrm{180C4F}^{171}$, $\mathrm{180F23}^{171}$, $\mathrm{256F}^{57}$, $\mathrm{297B}^{57}$, $\mathrm{37A5}^{57}$, $\mathrm{4B75}^{57}$, $\mathrm{1805AD}^{57}$, $\mathrm{180AD3}^{57}$, $\mathrm{180CB5}^{57}$, $\mathrm{1826A3}^{57}$, $\mathrm{183159}^{57}$
$C(22,5,3) \leq 176$ $[\mathbb{Z}_{22}]$ (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21)	$\mathrm{709}^{22}$, $\mathrm{923}^{22}$, $\mathrm{101B}^{22}$, $\mathrm{1285}^{22}$, $\mathrm{2A41}^{22}$, $\mathrm{5141}^{22}$, $\mathrm{8311}^{22}$, $\mathrm{8443}^{22}$
$C(23,5,3) \leq 187$ $[\mathbb{Z}_{22}]$ See $C(22,5,3)$	$\mathrm{14B}^{22}$, $\mathrm{661}^{22}$, $\mathrm{20D1}^{22}$, $\mathrm{4115}^{22}$, $\mathrm{7021}^{22}$, $\mathrm{9089}^{22}$, $\mathrm{401803}^{11}$, $\mathrm{402805}^{11}$, $\mathrm{404809}^{11}$, $\mathrm{408811}^{11}$, $\mathrm{410821}^{11}$
$C(23,9,7) \leq 11891$ $[F_{23,22}]$ (1,5,2,10,4,20,8,17,16,11,9,22,18,21,13,19,3,15,6,7,12,14), (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)	$\mathrm{12FB}^{506}$, $\mathrm{259F}^{506}$, $\mathrm{26D7}^{506}$, $\mathrm{2E37}^{506}$, $\mathrm{354F}^{506}$, $\mathrm{3717}^{506}$, $\mathrm{56CD}^{506}$, $\mathrm{5C5B}^{506}$, $\mathrm{5DA3}^{506}$, $\mathrm{6B47}^{506}$, $\mathrm{8737}^{506}$, $\mathrm{89CF}^{506}$, $\mathrm{91E7}^{506}$, $\mathrm{B257}^{506}$, $\mathrm{1067D}^{506}$, $\mathrm{5FD}^{253}$, $\mathrm{7AF}^{253}$, $\mathrm{13F9}^{253}$, $\mathrm{1B5B}^{253}$, $\mathrm{1CE7}^{253}$, $\mathrm{1D57}^{253}$, $\mathrm{1E4F}^{253}$, $\mathrm{47F1}^{253}$, $\mathrm{4EB9}^{253}$, $\mathrm{53E5}^{253}$, $\mathrm{55D5}^{253}$, $\mathrm{59CD}^{253}$, $\mathrm{66B3}^{253}$, $\mathrm{6AAB}^{253}$, $\mathrm{72A7}^{253}$, $\mathrm{102FD}^{253}$, $\mathrm{13559}^{253}$
$C(24,9,7) \leq 17457$ $[F_{23,22}]$ See $C(23,9,7)$	$\mathrm{15E7}^{506}$, $\mathrm{1C6F}^{506}$, $\mathrm{2797}^{506}$, $\mathrm{2B5B}^{506}$, $\mathrm{2EAB}^{506}$, $\mathrm{364F}^{506}$, $\mathrm{459F}^{506}$, $\mathrm{4B37}^{506}$, $\mathrm{4B9B}^{506}$, $\mathrm{4BE3}^{506}$, $\mathrm{5337}^{506}$, $\mathrm{565D}^{506}$, $\mathrm{5AD3}^{506}$, $\mathrm{6A4F}^{506}$, $\mathrm{8B4F}^{506}$, $\mathrm{97C3}^{506}$, $\mathrm{9AE3}^{506}$, $\mathrm{A4D7}^{506}$, $\mathrm{80027F}^{506}$, $\mathrm{800AED}^{506}$, $\mathrm{800CDB}^{506}$, $\mathrm{801657}^{506}$, $\mathrm{801A3B}^{506}$, $\mathrm{80232F}^{506}$, $\mathrm{802575}^{506}$, $\mathrm{802DC3}^{506}$, $\mathrm{8034C7}^{506}$, $\mathrm{80417B}^{506}$, $\mathrm{8041AF}^{506}$, $\mathrm{8044F5}^{506}$, $\mathrm{8062AB}^{506}$, $\mathrm{808177}^{506}$, $\mathrm{5FD}^{253}$, $\mathrm{6FB}^{253}$, $\mathrm{103B7}^{253}$, $\mathrm{117D1}^{253}$, $\mathrm{8003CF}^{253}$
$C(25,7,5) \leq 3600$ $[F_{25,4}]$ (1,8,2,11,4,22,3,19)(5,13,10,21,20,17,15,9)(6,16,12,7,24,14,18,23), (0,1,2,3,4)(5,6,7,8,9)(10,11,12,13,14)(15,16,17,18,19)(20,21,22,23,24)	CAB^{200}, $\mathrm{D3A}^{200}$, $\mathrm{158E}^{200}$, $\mathrm{171C}^{200}$, $\mathrm{86B1}^{200}$, $\mathrm{870B}^{200}$, $\mathrm{8A4E}^{200}$, $\mathrm{90C7}^{200}$, $\mathrm{A274}^{200}$, $\mathrm{1088B4}^{200}$, $\mathrm{108D84}^{200}$, $\mathrm{108E28}^{200}$, $\mathrm{7F0}^{100}$, $\mathrm{CE6}^{100}$, $\mathrm{DD8}^{100}$, $\mathrm{F2C}^{100}$, $\mathrm{F83}^{100}$, $\mathrm{15A9}^{100}$, $\mathrm{1745}^{100}$, $\mathrm{1E0E}^{100}$, $\mathrm{108525}^{100}$, $\mathrm{1091A8}^{100}$, $\mathrm{10C1A2}^{100}$, $\mathrm{140A26}^{100}$

Table 3
New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(25,8,6) \leq 10350$ $[F_{25,12}]$ (1,7,17,3,16,21,4,23,13, 2,14,9)(5,8,24,15,19,12, 20,22,6,10,11,18), (0,1, 2,3,4)(5,6,7,8,9)(10,11, 12,13,14)(15,16,17,18, 19)(20,21,22,23,24)	C3F^{300}, 1D55^{300}, 2D1D^{300}, 2D96^{300}, 8CBC^{300}, 91D6^{300}, 931D^{300}, 9C4E^{300}, 9C71^{300}, 9D29^{300}, A363^{300}, A71C^{300}, A8DA^{300}, AF30^{300}, B253^{300}, B8A5^{300}, C34E^{300}, C479^{300}, C58D^{300}, 18E23^{300}, 1C62A^{300}, 108AA6^{300}, 10910F^{300}, 10A623^{300}, 11A4A8^{300}, F78^{150}, FA6^{150}, 2C6B^{150}, 2E55^{150}, 9CE4^{150}, 9F90^{150}, AD62^{150}, B6A4^{150}, B6D0^{150}, BB22^{150}, CDC4^{150}, CE61^{150}, D561^{150}, D5A4^{150}, D748^{150}, DAC8^{150}, E728^{150}, 194A6^{150}, 1B226^{150}
$C(25,9,7) \leq 23700$ $[F_{25,12}]$ See $C(25,8,6)$	F3B^{300}, 16BD^{300}, 17D6^{300}, 1E2F^{300}, 8AD7^{300}, 8D76^{300}, 8E9B^{300}, 95AE^{300}, 95D9^{300}, 965D^{300}, 9A6D^{300}, 9B95^{300}, A0F7^{300}, A1DB^{300}, A4FA^{300}, A54F^{300}, A57C^{300}, A7B4^{300}, AACE^{300}, AE6A^{300}, AED1^{300}, B5D2^{300}, B8E3^{300}, B98E^{300}, B9C6^{300}, BCB2^{300}, BD07^{300}, C66E^{300}, CB3C^{300}, CFC1^{300}, D4F2^{300}, D674^{300}, D8B5^{300}, D8D3^{300}, D94E^{300}, DD58^{300}, DF09^{300}, E25E^{300}, E4BC^{300}, E9D8^{300}, F1B4^{300}, F29C^{300}, 18EA3^{300}, 19987^{300}, 1B196^{300}, 1B685^{300}, 1E31A^{300}, 1E349^{300}, 10846F^{300}, 1084F5^{300}, 1090D7^{300}, 109C65^{300}, 10ACA3^{300}, 10B463^{300}, 10C52B^{300}, 118D25^{300}, 1194B1^{300}, 11C4A5^{300}, FE6^{150}, FF8^{150}, 17E5^{150}, 17EA^{150}, 1DB9^{150}, 1ECE^{150}, 1F47^{150}, 2CF6^{150}, 2DB5^{150}, 2EBA^{150}, 2ECD^{150}, 2F8B^{150}, 3C3D^{150}, 3C57^{150}, 3C9E^{150}, 1085AD^{150}, 108673^{150}, 1086AB^{150}, 108727^{150}, 10885F^{150}, 108BF0^{150}, 10A665^{150}, 10A9B4^{150}, 10C21F^{150}, 10C3E2^{150}, 10E1A6^{150}, 10E266^{150}, 110D63^{150}, 1110E7^{150}, 1111B5^{150}, 1112CB^{150}, 11221F^{150}, 11256A^{150}, 112969^{150}, 118C4B^{150}, 118E31^{150}, 11C471^{150}, 1218E3^{150}, 121B43^{150}, 1240FC^{150}, 124279^{150}, 12D149^{150}

Table 3
New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(25,10,8) \leq 48576$ $[F_{23,11}]$ (1,2,4,8,16,9,18,13,3,6, 12)(5,10,20,17,11,22,21, 19,15,7,14), (0,1,2,3,4,5, 6,7,8,9,10,11,12,13,14, 15,16,17,18,19,20,21,22)	15DF^{253}, 1DF9^{253}, 1FAB^{253}, 27AF^{253}, 28FF^{253}, 2F6D^{253}, 2FD3^{253}, 356F^{253}, 376B^{253}, 37CD^{253}, 39CF^{253}, 3BD5^{253}, 4B6F^{253}, 4FE3^{253}, 59DB^{253}, 5E57^{253}, 5E9B^{253}, 647F^{253}, 69B7^{253}, 6B3D^{253}, 6CCF^{253}, 6DE5^{253}, 74BB^{253}, 7557^{253}, 7D1D^{253}, 7F91^{253}, 86FD^{253}, 8B77^{253}, 9D67^{253}, 9F2D^{253}, A35F^{253}, AC7D^{253}, AD3D^{253}, B2AF^{253}, B759^{253}, BEB1^{253}, C3F3^{253}, C5D7^{253}, CE6D^{253}, CEAB^{253}, D357^{253}, D3A7^{253}, D4F5^{253}, E3D9^{253}, EEA3^{253}, F279^{253}, F98D^{253}, FCA9^{253}, FD43^{253}, 103DF^{253}, 10DF5^{253}, 10FB9^{253}, 11AEB^{253}, 11F59^{253}, 126F3^{253}, 13397^{253}, 13C8F^{253}, 152D7^{253}, 15379^{253}, 15B33^{253}, 15BE1^{253}, 162ED^{253}, 174E9^{253}, 19575^{253}, 1A7B1^{253}, 800B7D^{253}, 800CF7^{253}, 801B73^{253}, 801B8F^{253}, 801BE5^{253}, 801F39^{253}, 80295F^{253}, 802CBB^{253}, 802E37^{253}, 802E5D^{253}, 8033D3^{253}, 8035AB^{253}, 80392F^{253}, 8039B9^{253}, 803AD9^{253}, 803B71^{253}, 8041FD^{253}, 80465F^{253}, 8046ED^{253}, 80479B^{253}, 804D97^{253}, 80533B^{253}, 80553D^{253}, 805727^{253}, 80574D^{253}, 8059D5^{253}, 80662F^{253}, 8066D5^{253}, 8067E1^{253}, 8073A5^{253}, 8075D1^{253}, 807A47^{253}, 807F21^{253}, 8085BB^{253}, 808E5B^{253}, 8092E7^{253}, 8098D7^{253}, 80A597^{253}, 80ADC3^{253}, 80B1D9^{253}, 80B547^{253}, 80C5F1^{253}, 80C91F^{253}, 80D25D^{253}, 80D671^{253}, 80ED89^{253}, 8102EF^{253}, 810BAD^{253}, 810BD9^{253}, 10006F7^{253}, 100073F^{253}, 1000D9F^{253}, 1000F67^{253}, 10016BB^{253}, 10018F7^{253}, 1001D6D^{253}, 1001EE3^{253}, 1001FB1^{253}, 10024FB^{253}, 10025DD^{253}, 10027E9^{253}, 1002A77^{253}, 1002BB5^{253}, 1002DA7^{253}, 1002DE5^{253}, 1003753^{253}, 10038AF^{253}, 1003A79^{253}, 1003C5D^{253}, 1003E33^{253}, 1004CBD^{253}, 100519F^{253}, 100529F^{253}, 10056C7^{253}, 1005DA5^{253}, 100658F^{253}, 1006AB9^{253}, 1006E4D^{253}, 10072A7^{253}, 1007467^{253}, 1007959^{253}, 100866F^{253}, 1008D5B^{253}, 100993D^{253}, 1009AA7^{253}, 1009E93^{253}, 100A4E7^{253}, 100B11F^{253}, 100C2F9^{253}, 100C3AD^{253}, 100EB25^{253}, 100F487^{253}, 1010B67^{253}, 1010CF9^{253}, 101134F^{253}, 10136A3^{253}, 10147B1^{253}, 18002FB^{253}, 180047F^{253}, 180059F^{253}, 18007A7^{253}, 1800B57^{253}, 1800B6B^{253}, 1800DCB^{253}, 1800FC5^{253}, 18013E5^{253}, 1801A4F^{253}, 1801CCD^{253}, 1801D33^{253}, 1801DD1^{253}, 18020F7^{253}, 18026AD^{253}, 1802CD5^{253}, 1802EA3^{253}, 18031B5^{253}, 1803475^{253}, 1803723^{253}, 1803A4B^{253}, 18042F3^{253}, 1804537^{253}, 1804759^{253}, 18049AD^{253}, 180546B^{253}, 18055C3^{253}, 1805995^{253}, 1806BA1^{253}, 1807913^{253}

Table 3

New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(26,8,6) \leq 13800$ $[F_{25,12}]$ See $C(25,8,6)$	779^{300}, 1717^{300}, 1D1E^{300}, 8DAC^{300}, 8F07^{300}, 938D^{300}, 946B^{300}, 9672^{300}, 9AB2^{300}, 9B64^{300}, 9CA6^{300}, 9F30^{300}, A0DE^{300}, A1F4^{300}, A527^{300}, A749^{300}, AA71^{300}, B3C2^{300}, B585^{300}, B5D0^{300}, B643^{300}, B86A^{300}, C5E1^{300}, C974^{300}, D25A^{300}, E315^{300}, F12C^{300}, F225^{300}, 10A04F^{300}, 128863^{300}, 2000CB9^{300}, 2000D66^{300}, 2000D9C^{300}, 20015B4^{300}, 200166A^{300}, 200168D^{300}, 2008475^{300}, 2008738^{300}, 2008786^{300}, 2009165^{300}, 2009447^{300}, 2009CA2^{300}, 200C236^{300}, 2D95^{150}, 2E55^{150}, AD62^{150}, DB41^{150}, 20007E4^{150}, 210842D^{150}
$C(26,9,7) \leq 33900$ $[F_{25,12}]$ See $C(25,8,6)$	1CDB^{300}, 1DB5^{300}, 1DD6^{300}, 1E9E^{300}, 847F^{300}, 895F^{300}, 89F6^{300}, 8A5F^{300}, 8EEC^{300}, 8FB8^{300}, 99CD^{300}, 99D3^{300}, 9B4E^{300}, 9C67^{300}, 9E59^{300}, 9F86^{300}, A376^{300}, A697^{300}, A72E^{300}, ADB4^{300}, B2CB^{300}, B69C^{300}, B987^{300}, B9E4^{300}, BAD2^{300}, C1DD^{300}, C1ED^{300}, C33E^{300}, C6BC^{300}, C8B7^{300}, CAF1^{300}, CB39^{300}, CD2E^{300}, CDF0^{300}, CF54^{300}, D1E3^{300}, D26D^{300}, D475^{300}, D59A^{300}, D786^{300}, DAAA^{300}, DB92^{300}, E52B^{300}, E725^{300}, EE68^{300}, EF03^{300}, F2A5^{300}, F4D4^{300}, F638^{300}, F872^{300}, 18F92^{300}, 19C72^{300}, 1AE34^{300}, 1C669^{300}, 10870F^{300}, 108CAD^{300}, 1091AD^{300}, 1093C3^{300}, 10A2A7^{300}, 10A2EA^{300}, 10ACAA^{300}, 10C2CE^{300}, 112179^{300}, 11246D^{300}, 1196C2^{300}, 11B628^{300}, 11C465^{300}, 2000775^{300}, 20007D3^{300}, 20015BC^{300}, 200845F^{300}, 2008679^{300}, 20086EA^{300}, 200899B^{300}, 2008E87^{300}, 2008F25^{300}, 200913B^{300}, 200939C^{300}, 20095C6^{300}, 2009770^{300}, 2009879^{300}, 2009AC3^{300}, 2009AE8^{300}, 2009B32^{300}, 2009D13^{300}, 200A2BA^{300}, 200A4AD^{300}, 200A939^{300}, 200AC71^{300}, 200B392^{300}, 200B4A3^{300}, 200C237^{300}, 200C31D^{300}, 200C6D4^{300}, 200E19C^{300}, 200E718^{300}, 200F066^{300}, 2108CA3^{300}, 210A1E4^{300}, 210B04B^{300}, 17E5^{150}, 1F47^{150}, 2ECD^{150}, 3D1B^{150}, 108575^{150}, 1086DA^{150}, 10909F^{150}, 1093E8^{150}, 10A11F^{150}, 10CA72^{150}, 10D26A^{150}, 10E1A6^{150}, 11616C^{150}, 1240FC^{150}, 2000DEC^{150}, 20016F2^{150}, 2001C73^{150}, 2002CB6^{150}, 2002CCD^{150}, 200AD62^{150}, 200D561^{150}, 200D5A4^{150}, 20194A6^{150}, 2019643^{150}, 2019A23^{150}, 2029865^{150}
$C(26,10,8) \leq 70000$ $[\mathrm{ASL}_2(5)]$ (0,19,13,18,4,5)(1,3,21, 17,15,22)(2,12,9,16,6, 14)(7,23,10,11,20,8), (1, 20,14,12,22,4,5,16,18,8) (2,15,23,24,19,3,10,7,6, 11)(9,21)(13,17)	1DE7^{3000}, 86FB^{3000}, 873F^{3000}, 95FA^{3000}, AD97^{3000}, BCEC^{3000}, BD5A^{3000}, BF8C^{3000}, 18FCA^{3000}, 195B5^{3000}, 196AB^{3000}, 19EB8^{3000}, 19F32^{3000}, 1093EC^{3000}, 20007EB^{3000}, 20014F7^{3000}, 2002C77^{3000}, 200867E^{3000}, 2008F2E^{3000}, 2009576^{3000}, 2009EA3^{3000}, 21085B3^{3000}, 19ED1^{1000}, 2009D4B^{1000}, 200AF19^{1000}, 2108A8F^{1000}

Table 3
New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(27,9,7) \leq 45279$ $[\mathrm{A\Gamma L}_1(27)]$ (0,1,2,3,4,5,6,7,8)(9,10, 11,12,13,14,15,16,17) (18,19,20,21,22,23,24, 25,26), (0,26,18,13,8,19) (1,16,20,15,6,5)(2,10,14, 23,3,11)(4,7,25,12,9,21) (22,24)	AFD^{2106}, $\mathrm{1B5D}^{2106}$, $\mathrm{1E5B}^{2106}$, $\mathrm{1EAB}^{2106}$, $\mathrm{2E79}^{2106}$, $\mathrm{2F5C}^{2106}$, $\mathrm{2FAA}^{2106}$, $\mathrm{3A9D}^{2106}$, $\mathrm{3B66}^{2106}$, $\mathrm{3F0B}^{2106}$, $\mathrm{4F3C}^{2106}$, $\mathrm{5769}^{2106}$, $\mathrm{5B56}^{2106}$, $\mathrm{66E6}^{2106}$, $\mathrm{6EF0}^{2106}$, $\mathrm{7715}^{2106}$, $\mathrm{9AAE}^{2106}$, $\mathrm{9F92}^{2106}$, $\mathrm{403E7}^{2106}$, $\mathrm{40BE9}^{2106}$, $\mathrm{76CA}^{1053}$, $\mathrm{9F25}^{1053}$, $\mathrm{1762C}^{1053}$
$C(27,10,8) \leq 97929$ $[\mathrm{A\Gamma L}_1(27)]$ See $C(27,9,7)$	$\mathrm{F3F}^{2106}$, $\mathrm{1BD7}^{2106}$, $\mathrm{1FAB}^{2106}$, $\mathrm{1FBC}^{2106}$, $\mathrm{23F7}^{2106}$, $\mathrm{2F6D}^{2106}$, $\mathrm{3F99}^{2106}$, $\mathrm{5EAD}^{2106}$, $\mathrm{5F65}^{2106}$, $\mathrm{6B6E}^{2106}$, $\mathrm{6B73}^{2106}$, $\mathrm{6EEC}^{2106}$, $\mathrm{6F5A}^{2106}$, $\mathrm{7796}^{2106}$, $\mathrm{77E4}^{2106}$, $\mathrm{7A3E}^{2106}$, $\mathrm{7BB4}^{2106}$, $\mathrm{7FC8}^{2106}$, $\mathrm{97D5}^{2106}$, $\mathrm{9E9E}^{2106}$, $\mathrm{AE79}^{2106}$, $\mathrm{AEE3}^{2106}$, $\mathrm{B6C7}^{2106}$, $\mathrm{B72E}^{2106}$, $\mathrm{BE6C}^{2106}$, $\mathrm{EE36}^{2106}$, $\mathrm{156AE}^{2106}$, $\mathrm{17738}^{2106}$, $\mathrm{1774A}^{2106}$, $\mathrm{17783}^{2106}$, $\mathrm{41E57}^{2106}$, $\mathrm{4353E}^{2106}$, $\mathrm{43A5B}^{2106}$, $\mathrm{43C2F}^{2106}$, $\mathrm{43D87}^{2106}$, $\mathrm{453DA}^{2106}$, $\mathrm{456CB}^{2106}$, $\mathrm{458EB}^{2106}$, $\mathrm{459F2}^{2106}$, $\mathrm{462CF}^{2106}$, $\mathrm{4741F}^{2106}$, $\mathrm{47F05}^{2106}$, $\mathrm{48BF4}^{2106}$, $\mathrm{49CA7}^{2106}$, $\mathrm{4A96D}^{2106}$, $\mathrm{927F}^{1053}$, $\mathrm{B6DA}^{1053}$, $\mathrm{EE8B}^{1053}$
$C(28,8,6) \leq 21762$ $[F_{27,26}]$ (1,3,9,5,15,23,13,17,20, 4,12,14,11,2,6,18,7,21, 16,26,22,10,8,24,25,19), (0,1,2)(3,4,5)(6,7,8)(9, 10,11)(12,13,14)(15,16, 17)(18,19,20)(21,22,23) (24,25,26)	$\mathrm{377}^{702}$, $\mathrm{17A6}^{702}$, $\mathrm{1BCA}^{702}$, $\mathrm{1F61}^{702}$, $\mathrm{3635}^{702}$, $\mathrm{39A5}^{702}$, $\mathrm{932E}^{702}$, $\mathrm{958B}^{702}$, $\mathrm{99C9}^{702}$, $\mathrm{9B52}^{702}$, $\mathrm{404F9}^{702}$, $\mathrm{407A5}^{702}$, $\mathrm{41559}^{702}$, $\mathrm{4172C}^{702}$, $\mathrm{41B23}^{702}$, $\mathrm{4207E}^{702}$, $\mathrm{42535}^{702}$, $\mathrm{42939}^{702}$, $\mathrm{43685}^{702}$, $\mathrm{44F0C}^{702}$, $\mathrm{461AC}^{702}$, $\mathrm{80012D5}^{702}$, $\mathrm{80013C3}^{702}$, $\mathrm{8001571}^{702}$, $\mathrm{800178A}^{702}$, $\mathrm{800730C}^{702}$, $\mathrm{8007321}^{702}$, $\mathrm{1E47}^{351}$, $\mathrm{36C3}^{351}$, $\mathrm{720F}^{351}$, $\mathrm{9669}^{351}$, $\mathrm{ADA1}^{351}$, $\mathrm{8040457}^{351}$, $\mathrm{804048F}^{351}$, $\mathrm{804099A}^{351}$
$C(28,9,7) \leq 62127$ $[\mathrm{A\Gamma L}_1(27)]$ See $C(27,9,7)$	AFD^{2106}, $\mathrm{FE6}^{2106}$, $\mathrm{1F6C}^{2106}$, $\mathrm{2ABB}^{2106}$, $\mathrm{46EB}^{2106}$, $\mathrm{57A6}^{2106}$, $\mathrm{5A4F}^{2106}$, $\mathrm{5AD6}^{2106}$, $\mathrm{5AF2}^{2106}$, $\mathrm{5BCC}^{2106}$, $\mathrm{5EC9}^{2106}$, $\mathrm{5F1A}^{2106}$, $\mathrm{665E}^{2106}$, $\mathrm{6E78}^{2106}$, $\mathrm{72D3}^{2106}$, $\mathrm{7B16}^{2106}$, $\mathrm{7B51}^{2106}$, $\mathrm{9B9A}^{2106}$, $\mathrm{ABF0}^{2106}$, $\mathrm{80003DB}^{2106}$, $\mathrm{800067E}^{2106}$, $\mathrm{80007E3}^{2106}$, $\mathrm{8001A9D}^{2106}$, $\mathrm{8001F29}^{2106}$, $\mathrm{8001F92}^{2106}$, $\mathrm{8002A37}^{2106}$, $\mathrm{8002B72}^{2106}$, $\mathrm{800674C}^{2106}$, $\mathrm{76CA}^{1053}$, $\mathrm{80012FC}^{1053}$, $\mathrm{8005E94}^{1053}$

Table 3
New covering designs (continued)

Bound [Group] Generators	Base blocks
$C(28,10,8) \leq 140049$ $[\mathrm{A\Gamma L}_1(27)]$ See $C(27,9,7)$	FCF^{2106}, $\mathrm{17F6}^{2106}$, $\mathrm{1F73}^{2106}$, $\mathrm{1FD5}^{2106}$, $\mathrm{27F6}^{2106}$, $\mathrm{3F5A}^{2106}$, $\mathrm{566F}^{2106}$, $\mathrm{5BCD}^{2106}$, $\mathrm{5F6C}^{2106}$, $\mathrm{5F87}^{2106}$, $\mathrm{5FB8}^{2106}$, $\mathrm{6AF9}^{2106}$, $\mathrm{6ED5}^{2106}$, $\mathrm{6F5C}^{2106}$, $\mathrm{72E7}^{2106}$, $\mathrm{73E3}^{2106}$, $\mathrm{77A5}^{2106}$, $\mathrm{77C9}^{2106}$, $\mathrm{7B66}^{2106}$, AABB^{2106}, AADD^{2106}, $\mathrm{AFC9}^{2106}$, $\mathrm{B7AC}^{2106}$, $\mathrm{BAF4}^{2106}$, $\mathrm{D7D4}^{2106}$, $\mathrm{DBE4}^{2106}$, $\mathrm{DE3A}^{2106}$, $\mathrm{F696}^{2106}$, $\mathrm{F7A2}^{2106}$, $\mathrm{FB61}^{2106}$, $\mathrm{429FC}^{2106}$, $\mathrm{4323F}^{2106}$, $\mathrm{4355E}^{2106}$, $\mathrm{43F0D}^{2106}$, $\mathrm{454FA}^{2106}$, $\mathrm{47ACA}^{2106}$, $\mathrm{47C56}^{2106}$, $\mathrm{48D3D}^{2106}$, $\mathrm{48EAD}^{2106}$, $\mathrm{49A5E}^{2106}$, $\mathrm{4AB59}^{2106}$, $\mathrm{8000F97}^{2106}$, $\mathrm{80017A7}^{2106}$, $\mathrm{8001B3D}^{2106}$, $\mathrm{8001B57}^{2106}$, $\mathrm{80023AF}^{2106}$, $\mathrm{80023DB}^{2106}$, $\mathrm{800276B}^{2106}$, $\mathrm{80033EC}^{2106}$, $\mathrm{80036E5}^{2106}$, $\mathrm{800379A}^{2106}$, $\mathrm{80037B4}^{2106}$, $\mathrm{8003A6E}^{2106}$, $\mathrm{80057CA}^{2106}$, $\mathrm{8005EC3}^{2106}$, $\mathrm{8005F1A}^{2106}$, $\mathrm{8005FC4}^{2106}$, $\mathrm{8006A37}^{2106}$, $\mathrm{8007A55}^{2106}$, $\mathrm{800970F}^{2106}$, $\mathrm{8009FA1}^{2106}$, $\mathrm{800AE65}^{2106}$, $\mathrm{800BF22}^{2106}$, $\mathrm{800EAF0}^{2106}$, $\mathrm{80411F9}^{2106}$, $\mathrm{80413DC}^{2106}$, $\mathrm{92FD}^{1053}$

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