Vertex cover on other graph ensembles: the effect of degree–degree correlations

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[a]Primary source: Alexei Vázquez and Martin Weigt, *Computational complexity arising from degree correlations in networks*, Physical Review E **67**, 027101 (2003).

- **Introduction**
- **Generalized correlated random graphs (GCRG's)**
- **Lattice gas on GCRG**
- **Numerics**
- **Applications**
- **Conclusions**

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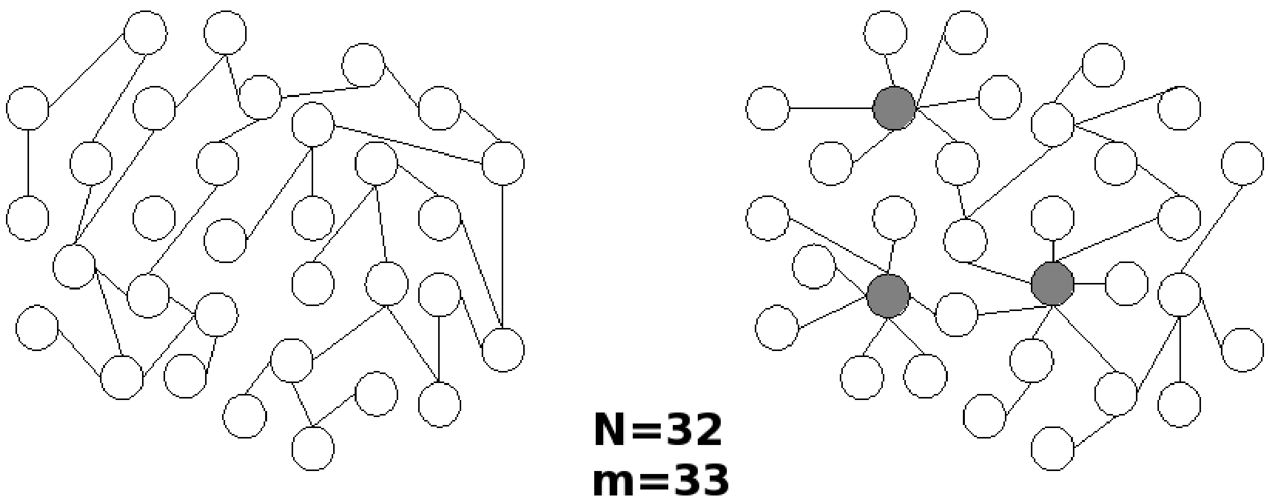
Introduction

- During the course we've seen VC's on Erdös-Renyi random graphs
- Real-worlds graphs are more complicated
  - Non-poissonian degree distributions (often fat tails)
  - Degree–degree correlations
- VC's on such graphs are important since they have applications in, for instance, network traffic monitoring



(a) Random Network **(b) Scale-free network**

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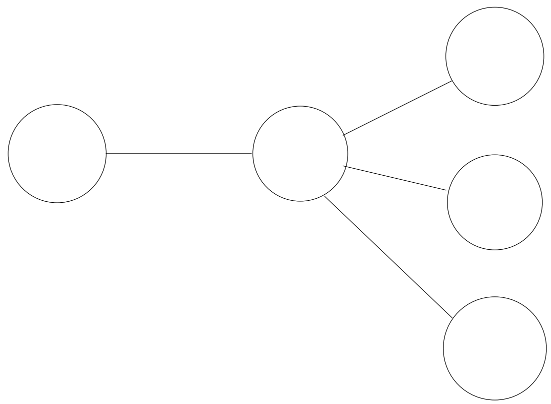
GCRG's

- A set on undirected graphs with N vertices and an arbitrary degree distribution p_d
- An important quantity is the excess degree distribution

$$q_d = \frac{(d+1)p_{d+1}}{\langle d \rangle} \qquad (1)$$



GCRG's (cont'd)

- Choose an edge randomly: the endpoints have excess degrees d and d' with probability

$$(2-\delta_{d,d'})e_{dd'} \tag{2}$$

- $e_{dd'}$ is related to the conditional probability that a vertex of degree d is reached coming from a vertex of degree d'

$$P(d|d') = \frac{e_{dd'}}{q_{d'}} \tag{3}$$

- For uncorrelated graphs $e_{dd'} = q_d q_{d'}$

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Lattice gas

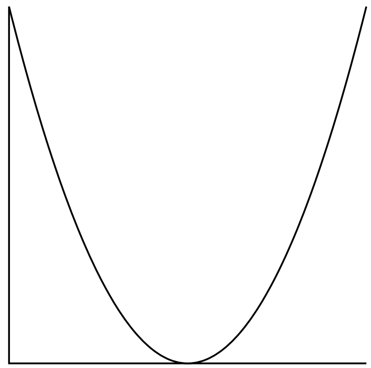
- Consider an arbitrary undirected graph with adjacency matrix J_{ij}
- A general lattice gas on the graph is defined by the Hamiltonian

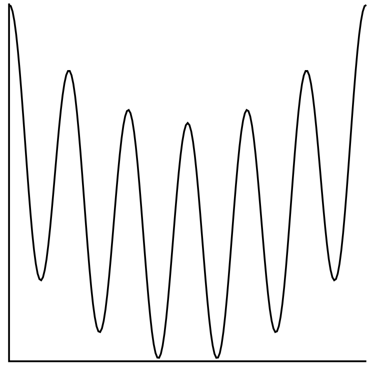
$$-\beta H = \sum_{i<j} J_{ij} w(x_i, x_j) + \mu \sum_i x_i \quad (4)$$

- The microscopic degrees of freedom $x_i = 0, 1$
- μ is the chemical potential
- The ferromagnetic Ising model is recovered by choosing $w(x_i, x_j) = (2x_i - 1)(2x_j - 1)$

Lattice gas (cont'd)

- Vázquez and Weigt perform a cavity calculation of the system
- In the end, it will be applied the VC's to estimate the relative size of the minimum vertex cover
- The calculation touches the issue of replica symmetry breaking without actually being an RSB calculation
- In this talk, I will next go through the main parts of the calculation



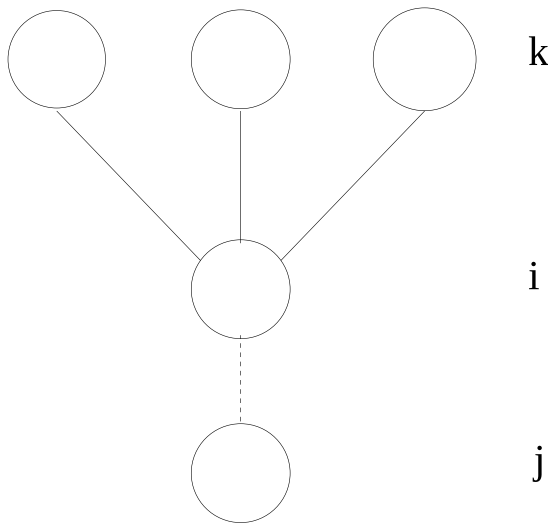


Partition functions

- Usually the partition function reads

$$Z = \sum_{\text{all states}} e^{-\beta H} \qquad (5)$$

- Assume the graph is locally treelike
- Now consider an arbitrary edge (i, j) and the subtree rooted in i with edge (i, j) removed

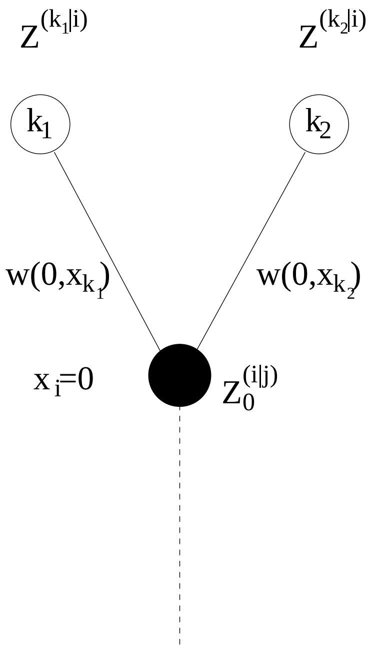


Partition functions (cont'd)

- Write down the partition functions with x_i fixed to value x

$$Z_0^{(i|j)} = \prod_{k \neq j | J_{ik}=1} \left(e^{w(0,0)} Z_0^{(k|i)} + e^{w(0,1)} Z_1^{(k|i)}\right)$$

$$Z_1^{(i|j)} = e^{\mu} \prod_{k \neq j | J_{ik}=1} \left(e^{w(1,0)} Z_0^{(k|i)} + e^{w(1,1)} Z_1^{(k|i)}\right)$$



Effective fields

- Define the effective fields as

$$h_{(i|j)} = \ln \frac{Z_1^{(i|j)}}{Z_0^{(i|j)}} \tag{6}$$

- Physical meaning: an isolated particle with $-\beta H = hx$, $Z_1 = e^h$, $Z_0 = 1$ and $h = \ln(Z_1/Z_0)$

- Using Eqs. from the previous slide we get

$$h_{(i|j)} = \mu + \sum_{k \neq j | J_{ik}=1} u(h_{k|i}) \tag{7}$$

$$u(h_{k|i}) = \ln(\frac{e^{w(1,0)} + e^{w(1,1)+h_{(k|i)}}}{e^{w(0,0)} + e^{w(0,1)+h_{(k|i)}}}) \tag{8}$$

Iteration and RS

- The equation for $h_{(i|j)}$ on the previous slide defines an iteration: for each step the h's can be substituted on the right-hand side, and new h's obtained
- The assumption that this iteration converges to a well-defined probability distribution $P(h)$ of the h's corresponds to the assumption that the replica symmetry is not broken.
- $P_d(h)$ for nodes of degree d is given by

$$P_d(h) = \int_{-\infty}^{\infty} \prod_{l=1}^{d} (dh_l \sum_{d'=0}^{\infty} p(d'|d) P_{d'}(h_l)) \delta(h - \mu - \sum_{l=1}^{d} u(h_l)) \quad (9)$$

Back to vertex covers

- The lattice gas is a vertex cover with the choice

$$e^{w(x_i,x_j)} = 1 - x_i x_j \tag{10}$$

- Here, $x_i = 0$ refers to a covered node and $x_i = 1$ to an uncovered one.
- The VC is minimum if the number of nodes with $x_i = 1$ is maximum.
- Therefore, take the limit $\mu \to \infty$ (scale the field by $h = \mu z$).

Back to vertex covers (cont'd)

- The equation for the probability distribution becomes

$$P_d(z) = \int_{-\infty}^{\infty} \prod_{l=1}^{d} (dz_l \sum_{d'=0}^{\infty} p(d'|d) P_{d'}(z_l)) \delta(h - \mu - \sum_{l=1}^{d} \max(0, z_l)) \tag{11}$$

- By a clever Ansatz this equation can be solved

The solution

- Omitting details, details, and details, one arrives at

$$\chi_c = 1 - \sum_{d=0}^{\infty} p_d (1 - \pi_{d-1})^{d-1} (1 + \frac{d-2}{2} \pi_{d-1}) \tag{12}$$

- The auxiliary variables π_d obey the self-consistency equation

$$\pi_d = \sum_{d_l=0}^{\infty} p(d_l|d)(1 - \pi_{d_l}^{d_l}) \tag{13}$$

- Physically π_d is the probability that an edge arriving at a vertex of degree $d+1$ carries a constraint, i.e. is not covered.

- This is “the same” as the iteration for the fields h but now for vertex classes.

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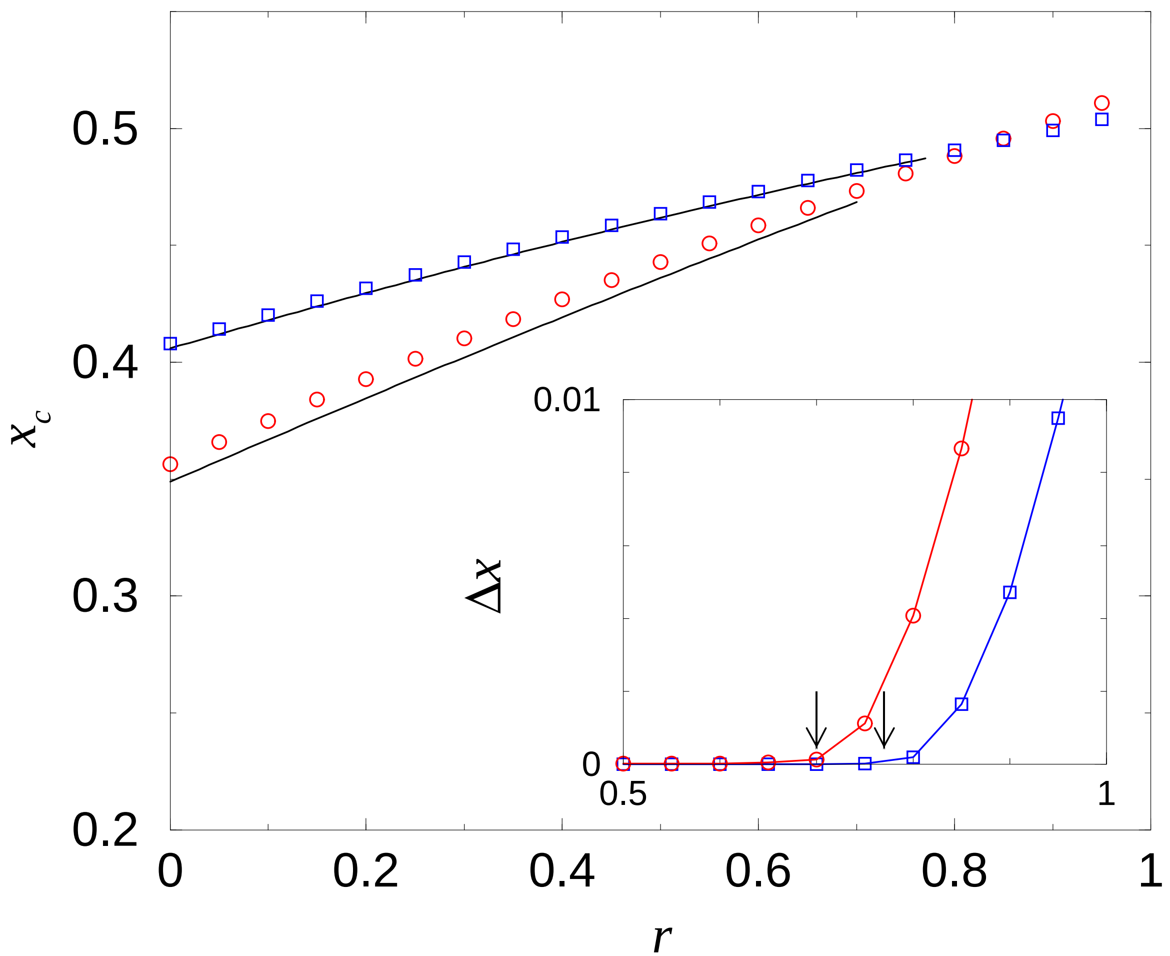
Numerics

- Protocol for solving χ_c
  - Iterate the equation for the π_d's
  - Convergence $\implies$ evaluate χ_c
  - Divergence $\implies$ RSB; no valid solution
- Compare this to what the leaf-removal algorithm gives for test graphs
  - Power-law degree distribution $p_d \propto d^{-\gamma}$ with $\gamma = 2.5$
  - Positive degree–degree correlations
    $e_{dd'} = q_d[r\delta_{d,d'} + (1-r)q_{d'}]$

Generating graphs

- Randomize degrees d_i independently for vertices using p_d's
- Create a set S of stubs such that each vertex i appears in it d_i times ($|S| = 2m$)
- For each edge, select first one endpoint randomly from S
- With probability r, select the other endpoint randomly from those with the same degree; otherwise randomly from all stubs
- Might lead to wanted correlations, but a bit suspicious (Catanzaro *et al.*, PRE **71**, 027103)



blue: $\gamma = 2.5$; red $\gamma = 3.0$; both $N = 10^6$

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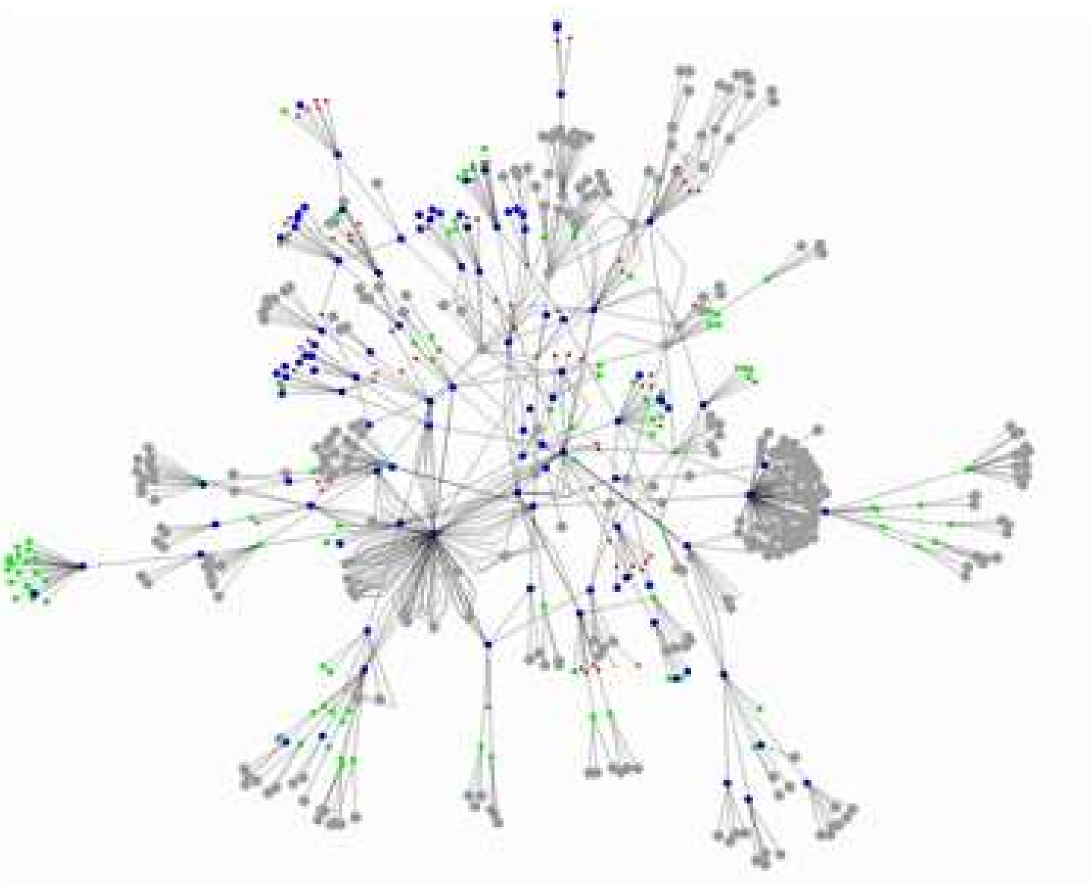
Applications

- Real-world networks come typically with two kinds of correlations: disassortative ($r < 0$) for technological nets and assortative ($r > 0$) for social networks (Newman, PRE $\mathbf{67}$, 026126 (2003)).
- These nets also have quite often fat tails, thus power-law is good as a rough approximation (Dorogovtsev and Mendes, Advances in Physics $\mathbf{51}$, 1079 (2002)).
- So, solving for the VC of a real technological net should be easy with the leaf-removal algorithm.
- This is of importance since the deployment of a network traffic monitoring system that is capable of observing all edges is essentially a vertex cover.



From GnuMap by Gregory Bray

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Conclusions

- There is an analytical replica-symmetric solution for the size of the minimum vertex cover in correlated random nets with arbitrary degree distribution
- This comes in the form of a self-consistency equation; convergence of the iteration in different cases reveals if RS holds or not
- Testing for prototype networks, positive correlations tend to break the replica symmetry
- VC's on real networks have applications in network traffic monitoring, for instance

Thanks for attention