Phase transitions in ground states of random-field systems and running time of maximum flow algorithm.

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October 14, 2007

Abstract

Random field Ising model (RFIM) is one of the simplest model with quenched randomness in physics. The ground-state of the RFIM can be solved in time, which scales polynomially with the system size. In this summary the critical slowing down of ground-state solving algorithm is considered based on work by A.A Middleton PRL, **88**,017202 (2002). A brief introduction to RFIM is given at the very begin after it maximum flow problem is introduced and algorithm, which can solve it push relabel algorithm is discussed, then the mapping on ground-state of RFIM to Maximum flow problem is made. At the end running times close to phase transition is considered.

1 Random field Ising model

The random field model was proposed originally by Larkin [1] to model the defect pinning of vortices in superconductors. The simplest version of the model for systems with discrete Ising symmetry is the RFIM. Fishman and Aharony [2] mapped the RFIM with a field of random sign and fixed magnitude with disordered bonds to an experimentally realizable system: the diluted antiferromagnet in a field (DAFF). RFIM is as well one of the simplest models with quenched randomness (i.e. random variables which do not evolve with time) and as such an important model on studies how the quenched randomness affects the behavior of spin systems.

The Hamiltonian of RFIM:

$$H = -J \sum_{\langle i,j \rangle} s_i s_j + \sum_i h_i s_i, \tag{1}$$

where J is the coupling constant sum is done over nearest neighbours, $s_i = \pm 1$ are Ising-type spins and h_i is the random field from Gaussian distribution with zero mean and variance Δ.

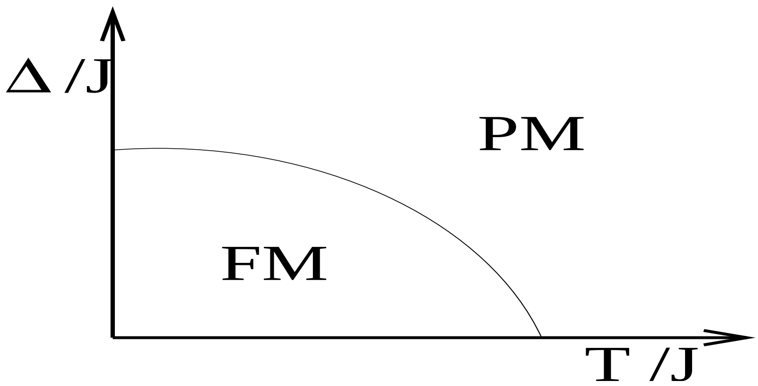

Figure 1: A phase diagram of three dimensional random field Ising model. Order parameter of this transition is the magnetization m (i.e. $m = \sum_i s_i$). A transition from high disorder paramagnetic state to ferromagnetic state is shown.

There is a phase transition in three dimensional random field Ising model. A phase diagram with paramagnetic and ferromagnetic phases are shown in Fig. 1. Generally in physics we are interested in the nature of phase transitions. A phase transition can be first or second order (i.e. if the first or second derivatives of the order parameter are discontinuous). In the case of second order phase transition different scaling exponents can be found when we are close to the phase transition. In physics we are interested in the behavior in the vicnicity of the phase transition:

$$\delta = \frac{\Delta - \Delta_c}{\Delta_c}, \tag{2}$$

where Δ_c is the variance of random field value in zero temperature phase transition boundary. One example on a scaling exponent is scaling exponent for correlation lenght ξ

$$\xi \sim |\delta|^{\nu}. \tag{3}$$

It is shown analytically that there is a phase transition in three dimensional random field Ising model [3]. A phase diagram has been studied using e.g. analytical studies [4], Monte Carlo method [5] and exact ground-state calculations [6]. A problem in ground state and Monte Carlo studies are the large finite size corrections needed for estimations of scaling exponents. Estimations from ground-state calculations in three dimensions for critical value of varince of random field in zero temperature is $\Delta_c = 2.27$ and the value for critical exponent $\nu = 1.37$ [6].

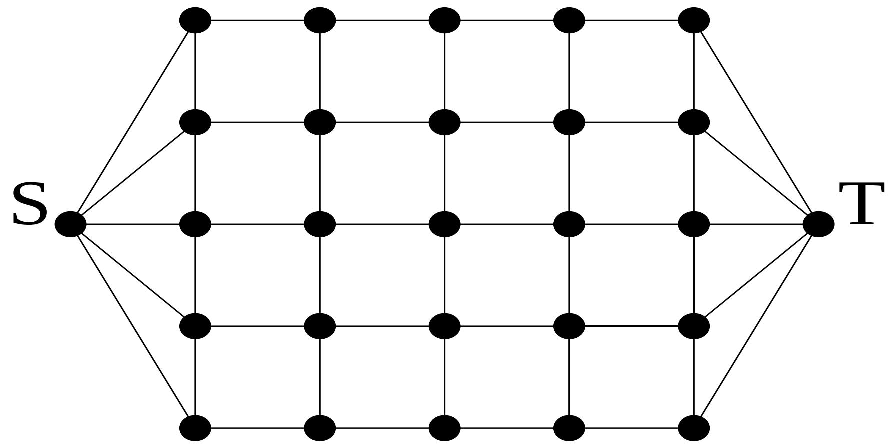

Figure 2: Maximum flow problem. The problem is to maximize the flow from source s to target t through the network.

The large energy barriers to equilibration make it very time consuming to sample configuration space accurately at low temperatures in system with quenches randomness. Finding the partition function for the RFIM at finite temperature is NP-hard [7]. As will be shown later chapters the ground-state of the RFIM can be solved with time, which scales only polynomially with the system size. This gives us possibility to study the zero-temperature phase transition with much larger system sizes than in the case with finite temperatures. This zero-temperature phase transition is expected to be in the same universality class (i.e. same scaling exponents) as the finite temperature transition [4]. This argument comes by assumption that in the long enough lenght-scales the randomness by random field is assumed to dominate over the randomness by temperature.

2 Maximum flow problem

The maximum flow problem is to find a feasible flow through a single-source, single-target flow network that is maximum. Concrete examples of such phenomenas are water flowing on the pipes or electricity moving on the electric circuit. This problem can be described by using following quantities: V, which contains all the vertices of the system, A contains all the edges of the system, capasities c between nodes, flow f between nodes, source s and target t. Capacity in our network means the maximum amount of flow, which can be transferred between i and j. Cut is a partition of the vertices of a graph into two sets S and $\bar{S}$. Many times in maximum flow problem we are interested in finding out the solution to min-cut problem at the same time. Min-cut is the bottleneck of the network (i.e. cut where $f_{ij} = c_{ij}$).

Menger's theorem states [8] that the maximum amount of flow is equal to the capacity of a minimal cut. More on this problem can be found e.g. [9].

3 Push relabel

Push relabel algorithm is effective algorithm for solving Max flow/Min cut problem. General running time of this algorithm scales as $O(V^2E)$. In this algorithm we need to measure two extra quantaties the excess:

$$e(i) = \sum_{\{j|(j,i)\in A\}} f_{ji} - \sum_{\{j|(j,i)\in A\}} f_{ij} \geq 0 \qquad \forall i \in V\backslash\{s,t\}, \tag{4}$$

and the distance $d(i)$, which starts with value of how many nodes away the node is from target and then evolve by relabel steps. The idea behind algorithm is to push excess to the direction of smaller distances.

The pseudoce of push-relabel algorithm:

algorithm preflow-push/relabel
begin
  preprocess
  while $e(i) > 0$ for any $i \in V\backslash\{s,t\}$ do
  begin
    choose $i \in V\backslash\{s,t\}$, with $e(i) > 0$
    push/relabel
  end
end

procedure preprocess
begin
  $f = 0$
  find exact $d(i)$ i.e. distance from target
  $f_{sj} = c_{sj}$ for $(s,j) \in A$, $e(j) = c_{sj}$
  $d(s) = \infty$
end
procedure push/relabel(i)
begin
  if the network contains $arc(i,j)$ with $d(i) = d(j) + 1$
    then push $min\{e(i), c_{ij} - f_{ij} + f_{ji}\}$ units of flow from node i to j
    else $d(i) = min_{j\in V}\{d(j) + 1|(i,j) \in A$ and $c_{ij} - f_{ij} + f_{ji} > 0$ }
end

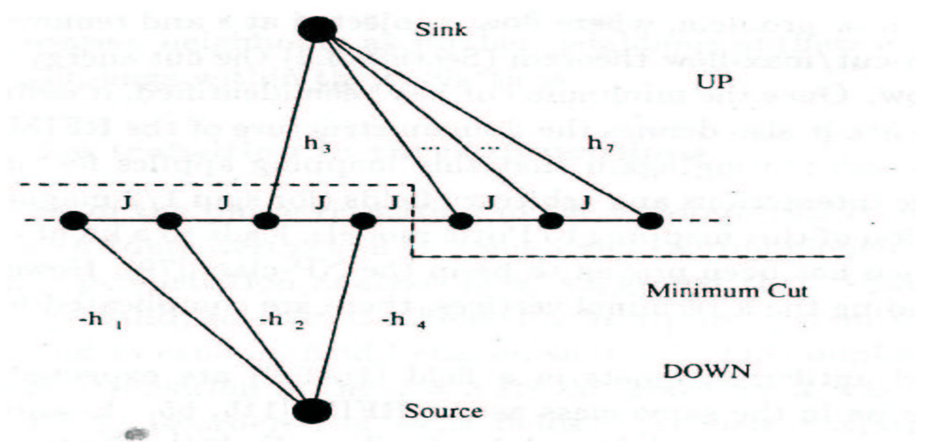

Figure 3: Example of RFIM ground-state mapping to maximum flow problem in one-dimensional random field Ising model. Minimum cut of the system is shown on broken line. Sink notation is used instead of target in figure

4 Mapping RFIM to Maximum flow problem.

The idea behind mapping RFIM to Maximum flow problem is that we add two new nodes to the system: the source s and the target t. The target node (or spin) takes the value $s_t = -1$ and the source spin take the value $s_s = -1$. Now we add new couplings between source/target and spins in the system with the following way:

All sites with positive random field are connected to sink s (and negative to target t):

$$J_{it} = \begin{cases} h_i & \text{if } h_i \geq 0 \\ 0 & \text{if } h_i < 0 \end{cases}$$

$$J_{is} = \begin{cases} 0 & \text{if } h_i \geq 0 \\ |h_i| & \text{if } h_i < 0 \end{cases}$$

Now our new RFIM Hamiltonian with two extra nodes is (with $s_s = -1$ and $s_t = 1$) :

$$H = -\sum_{(i,j)\in A} J_{ij} s_i s_j. \tag{5}$$

Energy of the Hamiltonian can be modelled by dividing the system to two energy terms:

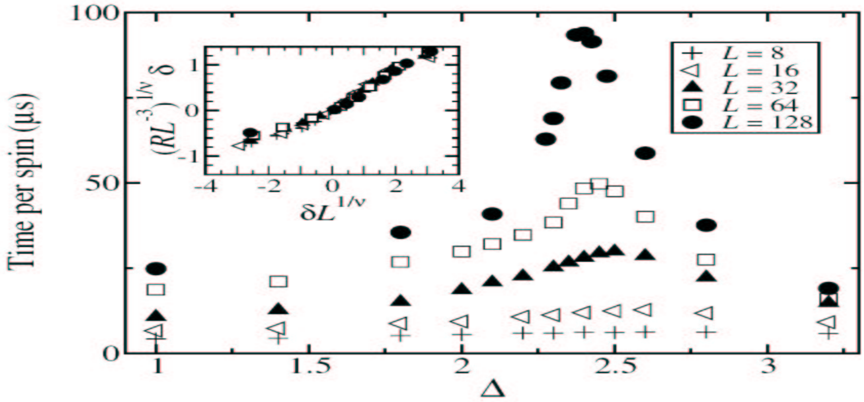

Figure 4: Critical slowing down of Middleton ground-state algorithm close to the phase transition in three dimension is shown ($\Delta_c = 2.27$, $\nu = 1.37$). Subfigure shows that the lower-bound estimate for running time of algorithm seems to work.

$$E(S) = -\sum_{(i,j)\in A} J_{ij} + 2\sum_{(i,j)\in(S,\bar{S})} J_{ij} = E_{bulk} + E_{dw}, \qquad (6)$$

where E_{bulk} is the energy when all of the minimum exchange and the field terms are satisfied and E_{dw} is there since not all couplings J_{ij} can be satisfied simultaneously. Now our problem is to find ground-state of RFIM (e.g. minimal cut on maximum flow problem).

5 Running times of Middleton algorithm

The algorithm Middleton [10] used in his study was a modification of push-relabel algorithm introduced earlier in this summary. In the variant Middleton used there was no source or target, but the algorithm used same ideas (i.e. flow, excess, distance). Node with the highest $d(i)$ was always chosen in their algorithm, and if set with the highest sites becomes isolated on their algorithm then all spins on set are connected to source. Global relabel updates were used in their algorithm as well time by time i.e. distance to target was calculated time by time as in preprocess in push-relabel algorithm.

Let us argue about the running time close to the phase transition (i.e. when $\Delta > \Delta_c$) The rearrangement of the spins scale with correlation lenght ξ of our system. By this we can get the lower-bound estimate for the relabel operations in our system. The size of distance difference (i.e. the number of totlal relabels) in nodes should be at least ξ in our system. For all L^d spins in our system we get the lower-bound estimate for the number of relabel operations: $L^d\xi$.

Fig. 4 shows running times of RFIM ground-state algorithm close to the phase transition in three dimensions. The figure shows the CPU time needed per spin on 766 Mhz PIII vs. disorder, with different values of disorder. The inset plot show that the arguments on scaling of running time $\Delta > \Delta_c$ case seems to be valid.

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