Cycles, Cuts and Spanning Trees

Sections 4 and 5 of “Algebraic Graph Theory” by N. Biggs

Leena Salmela

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Finite sets and vector spaces

- X is a finite set. The set of all functions from X to $\mathbb{C}$ has the structure of a finite dimensional vector space.
- If $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$, the vector space operations are defined as follows:

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x) \quad (x \in X, \alpha \in \mathbb{C})$$

- The dimension of the vector space is equal to the number of elements in X.

Vertex-space and edge-space

- Vertex-space $C_0(\Gamma)$: All functions from $V\Gamma$ to $\mathbb{C}$
  - $V\Gamma = \{v_1, v_2, \dots, v_n\}$ so the dimension is n.
  - Any function $\eta : V\Gamma \to \mathbb{C}$ can be represented as column vector $y = [y_1, y_2, \dots, y_n]^t$ where $y_i = \eta(v_i)$.
  - This corresponds to the standard basis $\{\omega_1, \omega_2, \dots, \omega_n\}$ defined by

$$\omega_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

- Edge-space $C_1(\Gamma)$: All functions from $E\Gamma$ to $\mathbb{C}$
  - Dimension is m.
  - The standard basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$:

$$\epsilon_i(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Orientation and incidence matrix

- For each edge one of the vertices is chosen to be the positive end and the other is chosen to be the negative end.
- Incidence matrix D of Γ with respect to the given orientation:

$$d_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j \\ -1, & \text{if } v_i \text{ is the negative end of } e_j \\ 0, & \text{otherwise} \end{cases}$$

- With respect to the standard bases D is a linear mapping from $C_1(\Gamma)$ to $C_0(\Gamma)$.
- The incidence matrix has rank $n - c$ where c is the number of connected components of Γ.

Cycles and the kernel of incidence mapping

- Definitions:
  - Rank of Γ: $r(\Gamma) = n - c$
  - Co-rank of Γ: $s(\Gamma) = m - n + c$
  - If Q is a subset of edges such that the subgraph $\langle Q \rangle$ is a cycle graph, then Q is a cycle in Γ. Two possible cyclic orderings of vertices in $\langle Q \rangle$ and thus two possible cycle-orientations. Define ξ_Q as follows:

$$\xi_Q(e) = \begin{cases} +1, & \text{if } e \text{ belongs to } Q \text{ and its orientation = its cycle-orientation.} \\ -1, & \text{if } e \text{ belongs to } Q \text{ and its orientation } \neq \text{ its cycle-orientation.} \\ 0, & \text{otherwise} \end{cases}$$

- The kernel of incidence mapping D of Γ is a vector space whose dimension is equal to co-rank of Γ.
- If Q is a cycle in Γ then ξ_Q belongs to the kernel of D.

Definition of cycle-subspace and cut-subspace

- Define an inner product between two elements ρ and σ of the edge-space of Γ:

$$(\rho, \sigma) = \sum_{e \in E\Gamma} \rho(e)\overline{\sigma(e)}$$

- The cycle-subspace of Γ is the kernel of the incidence mapping of Γ.
- The cut-subspace of Γ is the orthogonal complement of the cycle-subspace in $C_1(\Gamma)$ with respect to the inner product defined above.

About cuts and the cut-subspace

- Partition the vertices of Γ into two non-empty disjoint sets: $V = V_1 \cup V_2$.
- If the set of edges H which have one end in V_1 and other in V_2 is non-empty, then H is a cut in Γ.
- There are two possible orientations for the cut: Either V_1 contains all the positive ends and V_2 the negative ends or vice versa. Define ξ_H:

$$\xi_H(e) = \begin{cases} +1, & \text{if } e \text{ belongs to } H \text{ and its orientation = its cut-orientation.} \\ -1, & \text{if } e \text{ belongs to } H \text{ and its orientation } \neq \text{ its cut-orientation.} \\ 0, & \text{otherwise} \end{cases}$$

- The cut-subspace of Γ is a vector space whose dimension is equal to the rank of Γ (which is $n - c$). If H is a cut in Γ then ξ_H belongs to the cut-subspace.

The Laplacian matrix

D is the incidence matrix (with respect to some orientation) of a graph Γ and A is the adjacency matrix of Γ. Then the Laplacian matrix Q satisfies:

$$Q = DD^t = \Delta - A$$

where Δ is the diagonal matrix whose i:th diagonal entry is the degree of vertex v_i. Consequently Q is independent of the orientation given to Γ.

Spanning tree

- A spanning tree of Γ is a subgraph which has $n-1$ edges and contains no cycles.
- Let T be a spanning tree in a connected graph Γ:
  - For each edge g of Γ which is not in T there is a unique cycle $\mathrm{cyc}(T, g)$ in Γ containing g and edges in T only.
  - For each edge h of Γ which is in T there is a unique cut $\mathrm{cut}(T, h)$ in Γ containing h and edges not in T only.
- We give $\mathrm{cyc}(T, g)$ and $\mathrm{cut}(T, h)$ the orientation that coincides with the orientation of g and h in Γ.
- Then we have elements $\xi_{(T,g)}$ and $\xi_{(T,h)}$ which belong to the edge-space $C_1(\Gamma)$

Bases for the cycle-subspace and cut-subspace

- As g runs through the set $E\Gamma - T$, the $m - n + 1$ elements $\xi_{(T,g)}$ form a basis for the cycle-subspace of Γ.
- As h runs through the set T, the $n - 1$ elements $\xi_{(T,h)}$ form a basis for the cut-subspace of Γ.

Incidence matrix and spanning trees

- Any square submatrix of the incidence matrix D of Γ has determinant equal to 0 or +1 or -1.
- Let U be a subset of $E\Gamma$ with $|U| = n - 1$. Let D_U be the $(n-1) \times (n-1)$ submatrix of D consisting of those $n - 1$ columns that correspond to U and any $n - 1$ rows. Then D_U is invertible if and only if the subgraph $\langle U \rangle$ is a spanning tree of Γ.

Partitioning the incidence matrix

Label the edges so that edges belonging to a spanning tree T come first. The incidence matrix can then be partitioned as follows:

$$D = \begin{bmatrix} D_T & & D_N \\ & d_n & \end{bmatrix}$$

where D_T is an invertible $(n-1) \times (n-1)$ matrix and the last row d_n is linearly dependent on the other rows.

Basis for the cycle-subspace

- Let C be the matrix whose columns are the vectors representing elements $\xi_{(T,e_j)}(n \leq j \leq m)$ with respect to the standard basis of $C_1(\Gamma)$. Then

$$C = \begin{bmatrix} C_T \\ I_{m-n+1} \end{bmatrix}$$

- Each column of C is a cycle and thus belongs to the kernel of D so $DC = 0$ and furthermore:

$$C_T = -D_T^{-1} D_N$$

Basis for the cut-subspace

- Let K be the matrix whose columns represent the elements $\xi_{(T,e_j)}(1 \leq j \leq n-1)$. K can be written in the form:

$$K = \begin{bmatrix} I_{n-1} \\ K_T \end{bmatrix}$$

- Each column of K belongs to the orthogonal complement of the cycle-subspace so $K^t C = 0$. So $K_T^t + C_T = 0$ and

$$K_T = (D_T^{-1} D_N)^t$$

Application to electric networks (1)

- An electrical network is a connected graph Γ.
- The current and voltage vectors, w and z specify the physical characteristics of the network. These vectors belong to the edge-space.
- If M is a diagonal matrix whose entries are the conductances of the edges and n represents the externally applied voltages, then $z = Mw + n$.
- Kirchhoff's laws:

$$Dw = 0, \qquad C^t z = 0$$

- w and z can also be partitioned:

$$w = \begin{bmatrix} w_T \\ w_N \end{bmatrix}, \qquad z = \begin{bmatrix} z_T \\ z_N \end{bmatrix}$$

Application to electric networks (2)

- $Dw = 0$ gives $D_T w_T + D_N w_N = 0$ and since $C_T = -D_T^{-1} D_N$:

$$w_T = C_T w_N \qquad \text{and} \qquad w = C w_N$$

- So all the entries of the current vector are determined by the entries corresponding to edges not in T.
- Substituting in $z = Mw + n$ and premultiplying by C^t:

$$(C^t M C) w_N = -C^t n$$

- $C^t MC$ is invertible so this equation determines w_N and consequently both w and z.