Random Walks in Two Dimensions

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Examples

- Random walk in two dimensions. Escape routes and police. Figure 3.
- Voltage problem. Figure 4.

Harmonic Funcions in Two Dimensions

- $S = D \cup B$ is a set of lattice points in two dimensions. D are the interior points and B are the border points:
  - D and B have no points in common.
  - Every point in D has four neighboring points in S.
  - Every point in B has at least one of its neighboring points in D.
  - S hangs together in a nice way. Every point can be reached via a path from another point.
- Function f is harmonic if it has the averaging property for points (a, b) in D:

$$f(a,b) = \frac{f(a+1,b) + f(a-1,b) + f(a,b+1) + f(a,b-1)}{4}$$

Maximum and Uniqueness Principles

Maximum Principle:

- A harmonic function always attains its maximum (or minimum) on the boundary.

Uniqueness Principle:

- If $f(x)$ and $g(x)$ are harmonic functions such that $f(x) = g(x)$ in B then $f(x) = g(x)$ for all x.

The Dirichlet Problem

- Determine the two dimensional harmonic function when given the values of the function in the border.

The Monte Carlo Solution

- Simulate the random walk starting from all the interior points many times.
- For each x we can estimate the value of $f(x)$ by the average of simulations started at that point.
- This method is inefficient but somewhat colorful.

The Method of Relaxations

- Begin with any function having the specified border values.
- Run through the interior points and adjust their values.
- Repeat the previous step sufficiently many times.

Solution by Solving Linear Equations

- Write the equation that you get from the averaging property for each interior points.
- This set of equations can then be written in the form:

$$Ax = u$$

which can be solved by inversing the matrix A.

Finite Markov Chains

- There are a set $S = \{s_1, s_2, \ldots, s_r\}$ of states and a chance process moves around through these states.
- When the process is in state s_i it moves with probability P_{ij} to state s_j.
- The transition probabilities can be presented as a $r \times r$ matrix P called the transition matrix.
- In addition we specify a starting state for the chance process.

Absorbing and Non-Absorbing States

- A state that cannot be left once it is entered is called an absorbing state or a trap
- A Markov chain with at least one absorbing state is called absorbing.
- The states that are not traps are called non-absorbing.
- If a Markov chain is started at a non-absorbing state s_i we denote by B_{ij} the probability that the process will end up in s_j.

Properties of Markov Chains (1)

- Let P be the transition matrix of a Markov chain that has u absorbing states and v non-absorbing states. Let the states be ordered so that the absorbing ones come first. Then P can be presented as:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

- The matrix $N = (I - Q)^{-1}$ is called the fundamental matrix for the chain P.

- If 1 is a column vector of all ones then $t = N1$ gives the expected number of steps before absorption for each starting state.

Properties of Markov Chains (2)

- The absorption probabilities B are obtained from N by the matrix formula $B = NR$
- For an absorbing chain P the nth power P^n of the transition probabilities will approach

$$P^{\infty} = \begin{pmatrix} I & 0 \\ B & 0 \end{pmatrix}$$

Solution by the Method of Markov Chains (1)

- The random walk can be presented as a Markov chain: Each point is one state in the Markov chain and the transition matrix is defined based on the probabilities of going from one state to another.
- The border points of the random walk will be absorbing states and the interior points will be non-absorbing states.
- A function f is a harmonic function for a Markov chain P if

$$f(i) = \sum_j P_{ij} f(j)$$

- This is an extension of the averaging property.

Solution by the Method of Markov Chains (2)

- We write f as a column vector

$$f = \begin{pmatrix} f_B \\ f_D \end{pmatrix}$$

  where f_B are the values of f on the border and f_D are the values on the interior.

- From Markov chain theory we get

$$f_D = Bf_B$$

  where B_{ij} is the probability that starting from i the process will end up at j.

- Furthermore from Markov chain theory $B = NR = (I - Q)^{-1}R$