T-79.7001 Spring 2006: Algebraic and Spectral Graph Theory

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What is this?

Algebraic graph theory = study of graph properties via the algebraic characteristics of adjacency matrices

Spectral graph theory = algebraic graph theory specialised to eigenvalue characteristics

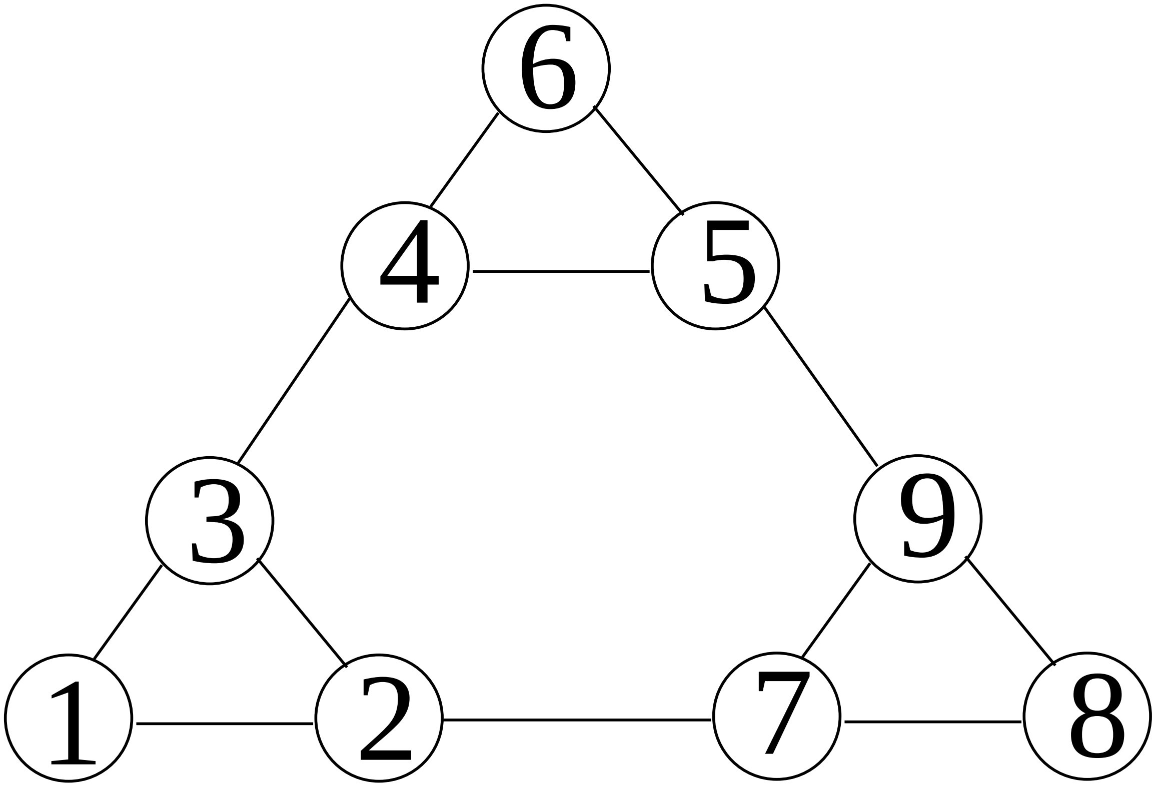
Why is this useful?

Algebraic features of adjacency matrices reflect graph properties in interesting & unexpected ways.

Provides deepened intuition into graphs, and makes possible the use of advanced tools from linear algebra, matrix theory and geometry in the study of graph properties.

Example: Paths & Connectivity

Consider the following graph G:



Example: Paths & Connectivity (cont'd)

This has adjacency matrix A_G:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Example: Paths & Connectivity (cont'd)

The square of this counts paths of length two in G:

$$A_G^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 3 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 3 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

In general, $A^k[i,j]$ equals the number of paths of length k from i to j.

Example: Paths & Connectivity (cont'd)

And of course the cube counts paths of length three:

$$A_G^3 = \begin{bmatrix} 2 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 5 & 1 & 2 & 1 & 5 & 1 & 1 \\ 4 & 5 & 2 & 5 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 5 & 2 & 5 & 4 & 2 & 1 & 1 \\ 1 & 2 & 1 & 5 & 2 & 4 & 1 & 1 & 5 \\ 1 & 1 & 1 & 4 & 4 & 2 & 1 & 1 & 1 \\ 1 & 5 & 1 & 2 & 1 & 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 2 & 4 \\ 1 & 1 & 2 & 1 & 5 & 1 & 5 & 4 & 2 \end{bmatrix}$$

In general, $A_G^k[i,j]$ = number of paths in G of length k from i to j.

Other simple properties

Let G be a connected graph with minimal degree δ and maximal degree Δ. Then:

- The maximal eigenvalue λ_0 of A_G satisfies $\delta \leq \lambda_0 \leq \Delta$.
- Every eigenvalue λ of A_G satisfies $|\lambda| \leq \Delta$.
- Δ is an eigenvalue of A_G if and only if G is regular.
- If Δ is an eigenvalue, then it has multiplicity 1. (For disconnected G, the multiplicity of Δ corresponds to the number of components.)
- If $-\Delta$ is an eigenvalue of A_G, then G is regular and bipartite.
- etc.

Example: Random walks

For a matrix P describing a random walk on a graph, the largest eigenvalue is always 1, with left eigenvector the stationary distribution π and right eigenvector $(1, 1, \ldots, 1)$. (Up to scaling.)

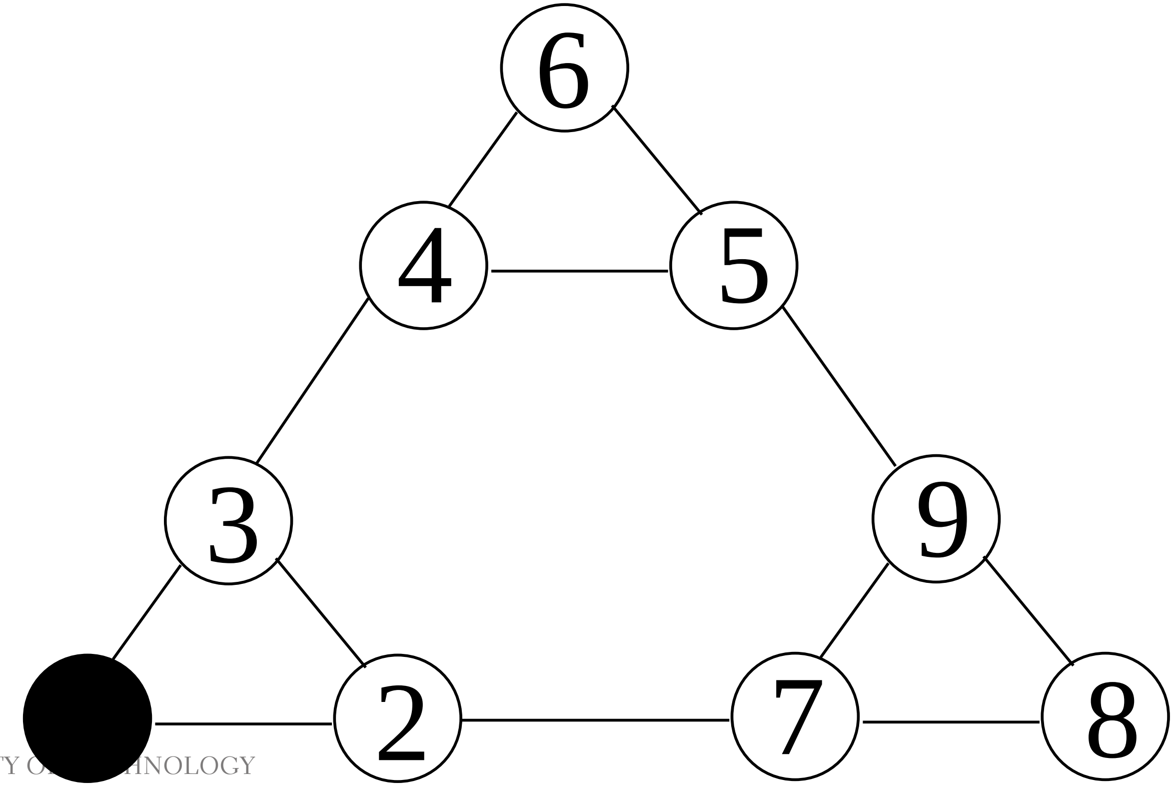
If the random walk is simple, i.e. at each node the choice of next neighbour is made uniformly at random, then all eigenvalues of P are real and satisfy: $1 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > -1$.

The convergence rate of the walk is dominated by the second largest eigenvalue modulus, which may be assumed to equal λ_1. (A simple modification to P will guarantee this.)

Application: Graph clustering

The right eigenvector associated to λ_1 is known as the *Fiedler vector* of the matrix. It provides information on the cluster structure of the graph.

Consider e.g. a random walk on the following graph G, which is otherwise simple but node 1 is absorbing:



Clustering (cont'd)

Then:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \end{bmatrix}$$

Clustering (cont'd)

In this case the eigenvalue spectrum is:

$$\lambda = [\ 1 \quad 0.95 \quad 0.72 \quad 0.40 \quad -0.13 \quad -0.20 \quad -0.48 \quad -0.59 \quad -0.67\ ],$$

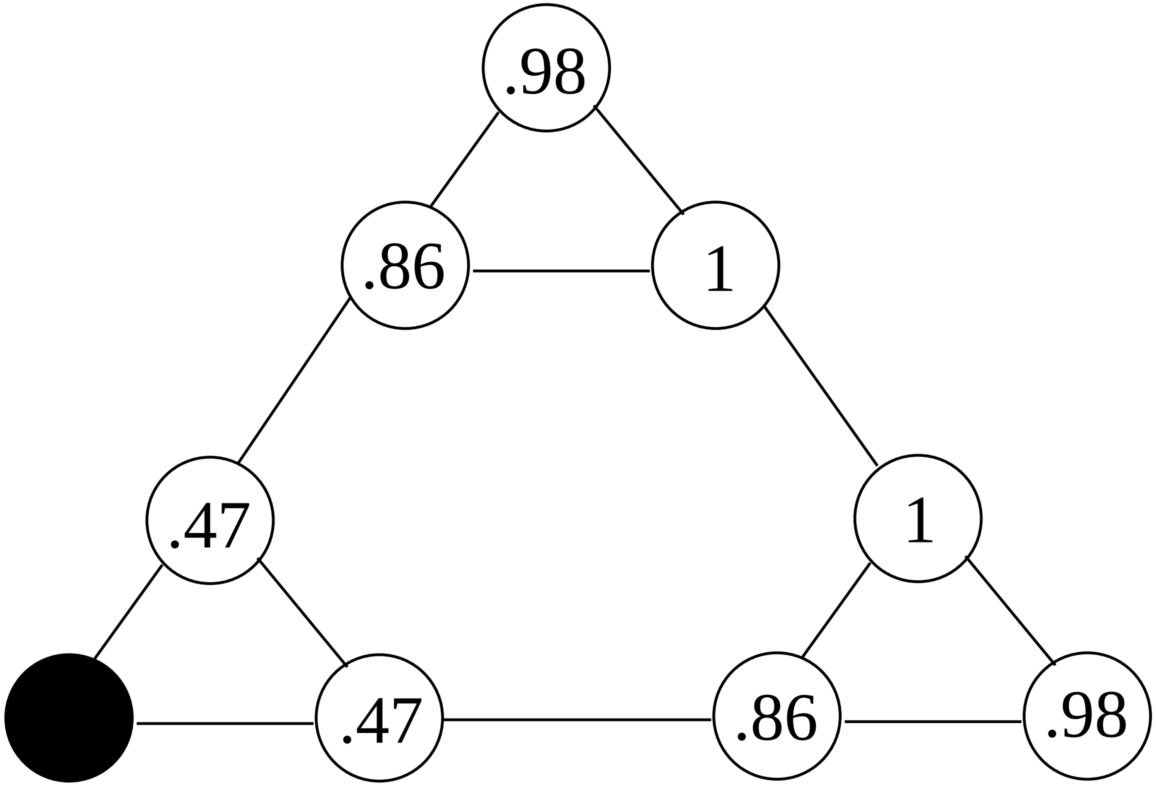
and the left and right eigenvectors corresponding to $\lambda_1 = 0.95$ are, suitably normalised:

$$\rho_1 = [\quad 1 \quad -0.08 \quad -0.08 \quad -0.14 \quad -0.17 \quad -0.11 \quad -0.14 \quad -0.11 \quad -0.17$$

$$u_1 = [\quad 0 \quad 0.47 \quad 0.47 \quad 0.86 \quad 1 \quad 0.98 \quad 0.86 \quad 0.98 \quad 1]$$

Small example (cont'd)

Superimposed on G, the Fiedler vector is:



The cluster of the absorbing node 1 is clearly discernible from the Fiedler values.