3. The Alteration Method

- Idea: Instead of proving the existence of the desired "good" structure directly, prove the existence of an "almost-good" structure and account for the corrections needed to make it "good".

- Example 1. Ramsey numbers.

Recall:

$R(k)$ = smallest n_k s.th
$n \geq n_k \rightarrow$ any two-colouring of edges of K_n contains a mono-χ K_k.

- Theorem 3.1 For any integer n,

$$R(k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Proof. For any given n, Consider a random (unif., indep.) two-colouring of K_n. For any subset of k vertices R, define indicator variable

$X_R \sim$ "R is monochromatic".

Then $X = \sum_R X_R$ counts the number of mono-χ K_k's and

$$E[X] = \sum_R E[X_R] = \binom{n}{k} \cdot 2^{1-\binom{k}{2}} \triangleq m.$$

Thus, for some two-colouring ξ of K_n, $X(\xi) \leq m$.

Now remove from K_n one vertex from each K_k that is mono-χ under ξ. Of course the same vertex may be marked for removal many times because of different K_k's, but in any case at most m vertices are removed. Now when restricted to the remaining K_s, $s \geq n-m$, colouring ξ contains no more mono-χ K_k's. $\square$

- Note. In the bound $g_k(n) = n - \binom{n}{k} 2^{1-\binom{k}{2}}$, $g_k(0)=0$, $g_k(n) \to -\infty$ as $n \to \infty$. An optimal choice of n is

$$n \sim \frac{1}{e} k 2^{k/2} (1 - o(1)),$$

yielding a bound $R(k) > \frac{1}{e}(1+o(1)) k 2^{k/2}$.

[Quick estimate: for $k \ll n$, $\binom{n}{k} \sim \left(\frac{ne}{k}\right)^k$, $2^{1-\binom{k}{2}} \sim 2^{-k^2/2}$.
Let's solve approximately for $g_k(n) > 0$:

$$\left(\frac{ne}{k}\right)^k \cdot 2^{-k^2/2} = 1 \Leftrightarrow \frac{ne}{k} \cdot 2^{-k/2} = 1 \Leftrightarrow n = \frac{k}{e} \cdot 2^{k/2}.$$]

- Also off-diagonal Ramsey numbers are often considered:

$R(k,\ell)$ = smallest $n_{k\ell}$ s.th
$n \geq n_{k\ell} \to$ any two-colouring (red/blue) of edges of K_n contains either a red K_k or a blue K_ℓ.

The basic probabilistic method (cf. Thm 1.1 or linearity of expectation) yields:

Theorem 3.2 If for some $p \in [0,1]$:

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$

then $R(k,\ell) > n$. □

By the alteration method, one can prove:

Theorem 3.3 For any integer n and $p \in [0,1]$:

$$R(k,\ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$$

Proof Consider random red-blue colouring of K_n. Compute exp. numbers of red K_k's and blue K_ℓ's. Remove one vertex from each, yielding a non-mono-χ colouring on a smaller K_s. □

- Example 2. Independent sets.

For a graph $G = (V, E)$, the independence number is

$$\alpha(G) = \max\{|S| : S \subseteq V, \text{ no two vertices in } S \text{ are connected by an edge in } E\}.$$

- Theorem 3.4 Let $G = (V, E)$, $|V| = n$, $d = \frac{2|E|}{n} \geq 1$ (avg degree). Then

$$\alpha(G) \geq \frac{n}{2d}.$$

Note This is one half of so called Turán's theorem (1941). The other half gives a structural characterisation of graphs with given $\alpha(G)$ and minimal number of edges (⟹ "extremal graph theory").

Proof. For a given (to be determined) $p \in [0, 1]$, consider a random (indep.) set of vertices $S \subseteq V$ defined by:

$$\Pr(v \in S) = p.$$

Assess the exp. number of vertices X and edges Y in S:

$$E[X] = np, \quad E[Y] = |E| \cdot p^2 = \frac{nd}{2} \cdot p^2.$$

Thus, by setting $p = \frac{1}{d}$:

$$E[X - Y] = np - \frac{nd}{2} p^2 = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d},$$

i.e. some $S \subseteq V$ contains $\geq \frac{n}{2d}$ many more vertices than edges. Removing one endpoint of each edge in S results in a set $S' \subseteq V$, $|S'| \geq \frac{n}{2d}$, where no two vertices are connected by an edge. $\square$

- Example 3. Discrepancy. (Uniform point placement.)

For a set S of n points in the unit square U, denote

$$T(S) = \text{min area of triangle spanned by three distinct points in } S$$

and

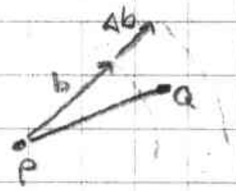
$$T(n) = \max\{T(S) : |S| = n\}.$$

- Heilbronn's conjecture (194?): $T(n) = O(\frac{1}{n^2})$.
- Disproved by Komlós, Pintz & Szemeredi (1982): $T(n) = \Omega_1(\frac{\log n}{n^2})$
- Theorem 3.5 (Erdös 1951). $T(n) \geq \frac{1}{100n^2} = \Omega_1(\frac{1}{n^2})$.

Proof. Consider a random (unif., indep.) set of points $P_1, P_2, \ldots, P_{2n}$ in the unit square, and compute the exp. number of triangles $P_i P_j P_k$ with area $\mu \leq 1/100n^2$.

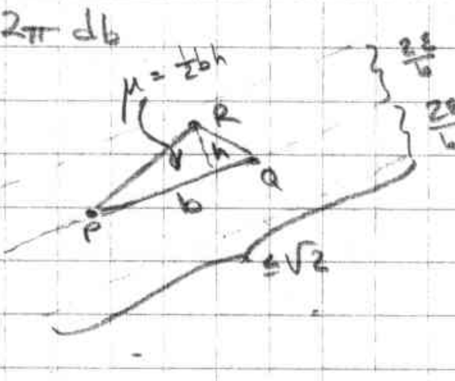
Calculation: for random points $P, Q, R \in U$ and given area ε, compute probability

$$\Pr(\mu = \mu(PQR) \leq \varepsilon).$$



1° Distribution of PQ distances:

$$\Pr(b \leq dist(P,Q) \leq b + \Delta b) = \pi(b+\Delta b)^2 - \pi b^2 = 2\pi \Delta b + (\Delta b)^2 \underset{\Delta b \to 0}{\longrightarrow} 2\pi\, db$$



2° Prob. of area ε, given base b:

$$\mu = \tfrac{1}{2} b \cdot h \leq \varepsilon \iff h \leq \frac{2\varepsilon}{b}$$

$$\Rightarrow \Pr(\mu \leq \varepsilon \mid dist = b) \leq \sqrt{2} \cdot 2 \cdot \frac{2\varepsilon}{b}$$

Thus,

$$\Pr(\mu(PQR) \leq \varepsilon) \leq \int_0^{\sqrt{2}} \Pr(\mu \leq \varepsilon \mid dist = b) \Pr(dist = b)\, db$$
$$= \int_0^{\sqrt{2}} 4\sqrt{2}\, \frac{\varepsilon}{b} \cdot 2\pi b \cdot db$$
$$= 8\sqrt{2}\pi\varepsilon \int_0^{\sqrt{2}} 1\, db$$
$$= \underline{\underline{16\pi\varepsilon}}.$$

- Hence for $\varepsilon = 1/100n^2$, $\Pr\left(\mu(PQR) \leq \frac{1}{100n^2}\right) < 0{,}6\, n^{-2}$ and

$$E\left[\# PQR, \mu \leq \frac{1}{100n^2}\right] < \binom{2n}{3} \cdot 0{,}6 n^{-2} < n.$$

- Consequently there is <u>some</u> placement of $2n$ points s.th. these induce <u>fewer</u> than n triangles of area $< 1/100n^2$.

Removing one point from each such small triangle results in a placement of $\geq n$ points that induce <u>no</u> triangles of area $< 1/100n^2$.

Hence $T(n) \geq \frac{1}{100n^2}$. $\square$

• Example 4. Volume packing.

Let $C_1 \subseteq \mathbb{R}^d$ be a bounded and convex set, centrally symmetric around the origin. Denote $\mu(C_1)$ = volume of C_1.

Denote

$N(C_1, x)$ = largest number of disjoint copies (translates) of C_1 that can be packed in $B(x) = [0,x]^d$.

Define the packing constant (or p. ratio) for C_1 as:

$$\delta(C_1) = \lim_{x\to\infty} \frac{N(C_1,x)\mu(C_1)}{\mu(B(x))} = \mu(C_1) \lim_{x\to\infty} N(C_1,x)\cdot x^{-d}.$$

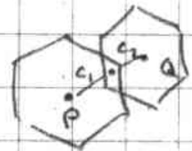
• Theorem 3.6 Under the above assumptions on C_1,

$$\delta(C_1) \geq 2^{-d-1}.$$

Proof. Consider a random (unif., indep.) set of n centerpoints $P_1, \ldots, P_n$ for copies of C_1 in $B(x)$, i.e. translates $P_1 + C_1, \ldots, P_n + C_1$ packed in $B(x)$.

For two random centerpoints P, Q, estimate the prob. of the overlap event $(P+C_1) \cap (Q+C_1) \neq \emptyset$.

This occurs iff there is some point $c = P + c_1 = Q + c_2$, i.e. if for some $c_1, c_2 \in C_1$:



$$P - Q = c_2 - c_1 = 2\,\frac{c_2 - c_1}{2} \in 2C_1$$

Event $P \in Q + 2C_1$ has prob. $\leq \mu(2C_1)x^{-d}$ for each Q, thus:

$$\Pr\left((P+C)\cap(Q+C) \neq \emptyset\right) \leq \mu(2C_1)x^{-d} = 2^d x^{-d} \mu(C_1).$$

For the random $P_1, \dots, P_n$, let X count the number of $i<j$ s.th. $(P_i + C) \cap (P_j + C) \neq \emptyset$. By lin. of exp.:

$$E[X] \leq \underbrace{\frac{n^2}{2} \cdot 2^d x^{-d} \mu(C_1)}_{g(x,n)}.$$

Hence there is some placement $P_1, \dots, P_n$ with $\leq g(x,n)$ overlapping pairs $P_i + C$, $P_j + C$. For each such pair, remove either P_i or P_j. This leaves $\geq n - g(x,n)$ nonoverlapping copies of C_1. Choose

$$n = x^d 2^{-d} / \mu(C_1)$$

to maximise this, yielding

$$N(C_1, x) \geq \frac{1}{2} x^d 2^{-d} / \mu(C),$$

i.e.

$$\delta(C_1) = \mu(C_1) \lim_{x \to \infty} N(C_1, x) x^{-d} \geq 2^{-d-1}.$$

(Strictly speaking, the packed C_1-copies may overflow the boundaries of $B(x)$, but since C_1 is bounded there is some constant w s.th. they lie in $B(x+w)$, and so

$$N(C_1, x+w) \geq \frac{1}{2} x^d 2^{-d} / \mu(C_1),$$

$$\delta(C_1) = \mu(C_1) \lim_{x+w \to \infty} N(C_1, x+w)(x+w)^{-d}$$

$$\geq 2^{-d-1} \lim_{x \to \infty} x^d (x+w)^{-d}$$

$$= 2^{-d-1}.) \quad \square$$

- Example 5. Girth and chromatic number.

- "Classic" result of Erdős (1959).

- For a graph $G = (V, E)$,
  - girth $\gamma(G)$ = length of shortest cycle
  - chromatic number $\chi(G)$ = min number of colours needed for vertex colouring so that no two adjacent vertices get the same colour

- Theorem 3.7 (Erdős 1959) For all k, ℓ there exists graph G with $\gamma(G) \geq \ell$, $\chi(G) \geq k$.

Proof. Fix $\theta < 1/\ell$ and consider a random graph $G \in \mathcal{G}(n, p)$, where $p = n^{\theta - 1}$. [n vertices, each edge present with prob. p, indep. of the others]

- Let X count the number of cycles of length $\leq \ell$ in G. Then:

$$E[X] = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i \leq \sum_{i=3}^{\ell} \frac{n^{\theta i}}{2i} = o(n). \quad \Big| \; \theta\ell < 1$$

It follows by Markov's inequality that [Markov: If $X \geq 0$, $a > 0$, then $\Pr(X \geq a) \leq E[X]/a$]

(1) $$\Pr(X \geq n/2) \leq \frac{E[X]}{n/2} = \underline{\underline{o(1)}}.$$

- Consider the independence number $\alpha(G)$ of G. For any r,

(2) $$\Pr(\alpha(G) \geq r) \leq \binom{n}{r}(1-p)^{\binom{r}{2}} < \left[n e^{-p(r-1)/2} \right]^r$$

By choosing $r = \left\lceil \frac{3}{p} \ln n \right\rceil$, also this is $\underline{\underline{o(1)}}$.

- Let n be so large that both probabilities (1) and (2) are $< 1/2$.

Then there is some G with n vertices, fewer than $n/2$ cycles of length $\le \ell$ and max indep. set size

$$\alpha(G) \le 3n^{1-\theta} \ln n.$$

- Remove from G a vertex from each cycle of length $\le \ell$. This gives a graph G^* with $\ge n/2$ vertices, girth $\gamma(G^*) > \ell$ and $\alpha(G^*) \le \alpha(G)$.

For the chromatic number of G^* the following holds:

$$\chi(G^*) \ge \frac{|G^*|}{\alpha(G^*)} \ge \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n} > k,$$

for sufficiently large n. $\square$