T-79.5201 Discrete Structures, Autumn 2007

Tutorial 8, 28 November

1. Let $n \geq 1$. A family $\mathcal{U}$ of subsets of $[n] = \{1, \ldots, n\}$ is an *upset* or *filter* if $A \in \mathcal{U}, A \subseteq B \implies B \in \mathcal{U}$. Similarly, $\mathcal{D} \subseteq \mathcal{P}([n])$ is a *downset* or *ideal* if $A \in \mathcal{D}, B \subseteq A \implies B \in \mathcal{D}$.

   Prove *Kleitman's Lemma:* If $\mathcal{U}$ is an upset on $[n]$ and $\mathcal{D}$ is a downset on $[n]$, then:

$$|\mathcal{U}| \cdot |\mathcal{D}| \geq 2^n |\mathcal{U} \cap \mathcal{D}|.$$

2. A family $\mathcal{A}$ of subsets of $[n]$ is *intersecting*, if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. Prove the following claims:

   (a) For any intersecting family $\mathcal{A}$ on $[n]$, $|\mathcal{A}| \leq 2^{n-1}$.

   (b) For any intersecting family $\mathcal{A}$ on $[n]$ that moreover satisfies $A \cup B \neq [n]$ for any $A, B \in \mathcal{A}$, $|\mathcal{A}| \leq 2^{n-2}$. (*Hint:* Observe that for any family of sets $\mathcal{A}$, there exist an upset $\mathcal{U}$ and a downset $\mathcal{D}$ such that $\mathcal{A} = \mathcal{U} \cap \mathcal{D}$.)

   (c) The bounds on the size of $\mathcal{A}$ in both (a) and (b) are best possible, i.e. there are families $\mathcal{A}$ that achieve these bounds.

3. [Alon & Spencer, Prob. 6.3:] Show that the probability that in a random graph $G \in \mathcal{G}(2k, 1/2)$, the maximum degree of any vertex is at most $k-1$, is at least $1/4^k$.