T-79.5201 Discrete Structures, Autumn 2007

Tutorial 4, 17 October

1. Let $k > 0$ be an integer, and let $p(n) \geq (6k \ln n)/n$ for large n. Prove that a.a.s. a random graph $G \in \mathcal{G}(n,p)$ contains no independent set of vertices of size $n/(2k)$, i.e.

$$\Pr\left(\alpha(G) \geq \frac{n}{2k}\right) \to 0 \quad \text{as } n \to \infty.$$

2. A *forest* is a graph with no nontrivial cycles (i.e. cycles of length ≥ 3). Prove that if $np \to 0$ as $n \to \infty$, then a.a.s. a random graph $G \in \mathcal{G}(n,p)$ is a forest. (Do this directly, by counting the expected number of cycles, without appealing to Theorem 4.6 in the lecture notes.)

3. Prove that $p(n) = \ln n/n$ is a threshold function for the disappearance of isolated vertices in a random graph $G \in \mathcal{G}(n,p)$. (*Hint:* Consider the random variable $X = X_1 + \cdots + X_n$, where X_i indicates whether vertex i is isolated in G. When estimating the variance of X, observe that $X_i X_j = 1$ iff both vertices are isolated. That requires forbidding $2(n-2)+1$ edges, so $E[X_i X_j] = (1-p)^{2n-3}$.)