T-79.5201 Discrete Structures, Autumn 2006

Tutorial 3, 11 October

1. Prove that if $a = \langle a_n \rangle$, $b = \langle b_n \rangle$ and $c = \langle c_n \rangle$ are sequences (of complex numbers) satisfying $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, then their exponential generating functions satisfy $\hat{c}(z) = \hat{a}(z)\hat{b}(z)$.

2. The *Bell number* b_n indicates how many different partitions (equivalence relations) can be constructed on a set of n elements. (In terms of Stirling numbers of the 2nd kind one could thus write $b_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$.) Show that the Bell numbers b_n satisfy the recurrence

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k, \quad b_0 = 1,$$

and derive from this the exponential generating function $\hat{b}(z)$ for the sequence $b = \langle b_n \rangle$. (*Hint:* Differentiate the series defining the sequence's egf and solve the resulting differential equation.)

3. Let $\mathcal{A} = (A, w_A)$ and $\mathcal{B} = (B, w_B)$ be two weighted families of combinatorial structures. Construct their *product* $\mathcal{A} \times \mathcal{B} = (C, w_C)$ by defining on the ambient set $C = A \times B$ the weight function as $w_C((\alpha, \beta)) = w_A(\alpha) + w_B(\beta)$. Prove that this product construction is ogf-admissible, i.e. that the ogf of the product family can be computed directly from the ogf's of the component families. (*Hint:* Look at the ogf sums as defined over the structures in each of the families, so. $c(z) = \sum_{\sigma \in C} z^{w_C(\sigma)} = \cdots$)