T-79.5201 Discrete Structures, Autumn 2006

Home assignment 1 (due 25 Oct at 12:15 p.m.)

1. Solve the following recurrence equations using the technique of generating functions:

   (a)

   $$\begin{cases} a_0 = 1, \\ a_n = 2a_{n-1} + n, \quad n \geq 1; \end{cases}$$

   (b)

   $$\begin{cases} s_0 = 0, \\ s_n = s_{n-1} + n^2, \quad n \geq 1. \end{cases}$$

2. (a) Consider the formal power series $F(X) = (\mathrm{Exp}(X)-1)/X = \sum_{n\geq 0} \frac{1}{(n+1)!} X^n$. Verify that this has an inverse $B(X) = X/(\mathrm{Exp}(X) - 1)$, and determine by formal expansion the coefficients $B_0, \ldots, B_4$ in the series $B(X) = \sum_{n\geq 0} \frac{B_n}{n!} X^n$.

   (b) The coefficients B_n determined in part (a) of the problem are called *Bernoulli numbers*. Show that they satisfy the recurrence equation

   $$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad n \geq 2.$$

   (*Hint:* Product formula for power series.)

3. An *involution* is a permutation that is its own inverse. (In terms of the cycle decomposition this means that the permutation only has cycles of lengths one and two.) Determine a simple recurrence formula for the number of involutions on n elements, and based on this the egf for the family of involutions. (*Hint:* Partition the involutions on the set $[n] = \{1, \ldots, n\}$ according to which other elements are included in the same cycle as element n.)

4. Determine the following ordinary generating functions, based directly on the structure of the respective combinatorial families:

   (a) The ogf for sequence $\langle a_n \rangle$, where a_n = the number of strings composed of digits 1 and 2, such that the digits add up to n. (By direct counting one observes that $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$ etc.)

   (b) The ogf for sequence $\langle b_n \rangle$, where b_n = the number of unlabeled rooted ordered trees with n nodes. (A tree is *rooted* if it has a distinct root node, and *ordered*, if the descendants of each node have a left-to-right ordering. In this case one obtains $b_0 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 2$, $b_4 = 4$ etc. *Hint:* Consider first the family of nonempty such trees.)