T-79.5102 **Autumn 2006**
Special Course in Computational Logic
Tutorial 2
Solutions

1. (a) $P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.200$

(b) $P(toothache|cavity) = \frac{P(toothache \wedge cavity)}{P(cavity)} = \frac{0.120}{0.200} = 0.6$

(c) $P(cavity) = 0.108 + 0.012 + 0.072 + 0.008 = 0.200$

(d) $P(cavity|toothache \vee catch) = \frac{P(cavity \wedge (toothache \vee catch))}{P(toothache \vee catch)}$
$= \frac{0.192}{0.108+0.012+0.016+0.064+0.072+0.144} \approx 0.462$

2. Let us the following propositions:

$$A = \text{“A person is ill”}$$
$$B = \text{“The test is positive”}$$

Since only one person in 10000 has the illness, the prior probability $P(A) = 1/10000$.

The test accuracy is 99% so the following conditional probabilities hold:

$$\begin{aligned} P(B \mid A) &= 0.99 \\ P(\neg B \mid \neg A) &= 0.99 \\ P(B \mid \neg A) &= 0.01 \\ P(\neg B \mid A) &= 0.01 \end{aligned}$$

There are two ways how the test may be positive: either the person is ill and the test gives the correct result, or the person is not ill but the test gives the wrong result. Since the two cases are mutually exclusive:

$$P(B) = P(B \mid A)P(A) + P(B \mid \neg A)P(\neg A)$$

We then use Bayes theorem to compute the probability that a person is ill given that the test result is positive:

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B)} \\ &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \neg A)P(\neg A)} \\ &= \frac{0.99 \cdot \frac{1}{10000}}{0.99 \cdot \frac{1}{10000} + 0.01 \cdot (1 - \frac{1}{10000})} \\ &= 0.0098 \end{aligned}$$

So, the patient does not have to worry, yet, since the probability of an incorrect diagnosis is much higher than of the illness.

3. Let us use the following lemmata (see slides for the derivations):

(i) $P(\phi \wedge \psi) = P(\phi) + P(\psi) - P(\phi \vee \psi)$.

(ii) $P(\neg\phi) = 1 - P(\phi)$.

(iii) $P(\phi) = P(\psi)$, if $\models \phi \leftrightarrow \psi$.

Using these, we obtain

$$
\begin{array}{lll}
 & P(A \leftrightarrow \neg B) & \\
= & P((A \vee B) \wedge \neg(A \wedge B)) & \text{(iii)} \\
= & P(A \vee B) + P(\neg(A \wedge B)) - P((A \vee B) \vee \neg(A \wedge B)) & \text{(i)} \\
= & P(A \vee B) + P(\neg(A \wedge B)) - 1 & \text{(A3)} \\
= & P(A) + P(B) - P(A \wedge B) + P(\neg(A \wedge B)) - 1 & \text{(A4)} \\
= & P(A) + P(B) - P(A \wedge B) + 1 - P(A \wedge B) - 1 & \text{(ii)} \\
= & P(A) + P(B) - 2 \times P(A \wedge B). &
\end{array}
$$

Note that $A \leftrightarrow \neg B$ expresses the *exclusive or* (xor) of A and B.

4. The problem is formalised using the following propositions:

$$
\begin{aligned}
A &= \text{“Prisoner } A \text{ is executed”} \\
B &= \text{“Prisoner } B \text{ is executed”} \\
C &= \text{“Prisoner } C \text{ is executed”}
\end{aligned}
$$

The propositions are mutually exclusive and each has a prior probability of $1/3$. We add the proposition:

$$D = \text{“The guard tells } B \text{ that he survives”}$$

After this the first proposition splits into two different cases: one where the guard tells B and one where he tells C. Since the guard does not lie, the other two propositions are not modified. Now:

$$
\begin{aligned}
P(\{A, D, \neg B, \neg C\}) &= \frac{1}{6} \\
P(\{A, \neg D, \neg B, \neg C\}) &= \frac{1}{6} \\
P(\{\neg A, \neg D, B, \neg C\}) &= \frac{1}{3} \\
P(\{\neg A, D, \neg B, C\}) &= \frac{1}{3}
\end{aligned}
$$

The guard tells A that B survives, so he now can compute his changes using the equation:

$$
\begin{aligned}
P(A \mid D) &= \frac{P(D \mid A)P(A)}{P(D)} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{1}{3}
\end{aligned}
$$