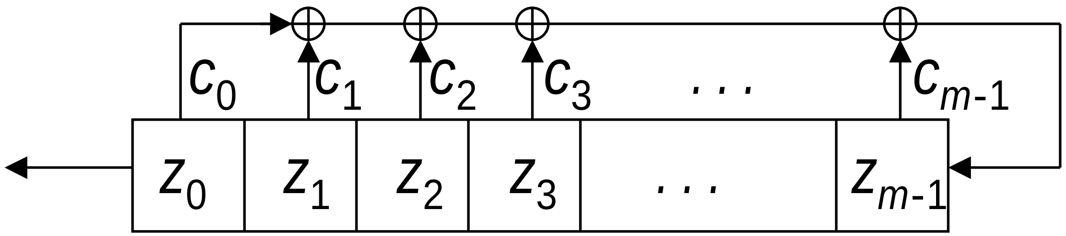
Linear Feedback Shift Registers 1/12

A binary linear feedback shift register (LFSR) is the following device



where the *i* th tap constant c_i = 1, if the switch connected, and c_i = 0 if it is open. The contents of the register $z_0, z_1, z_2, z_3, \ldots, z_{m-1}$ are binary values. Given this state of the device the output is z_0 and the new contents are $z_1, z_2, z_3, \ldots, z_{m-1}, z_m$, where z_m is computed using the recursion equation

$$z_m = c_0 z_0 + c_1 z_1 + c_2 z_2 + c_3 z_3 + \ldots + c_{m-1} z_{m-1}$$

The sum is computed *modulo* 2. As this process is iterated, the LFSR outputs a binary sequence $z_0, z_1, z_2, z_3, \ldots, z_{m-1}, z_m, \ldots$ Then the terms of this sequence satisfy the linear recursion relation

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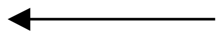
$$z_{k+m} = c_0 z_k + c_1 z_{k+1} + c_2 z_{k+2} + c_3 z_{k+3} + \ldots + c_{m-1} z_{k+m-1}$$

for all $k = 0,1,2,\ldots$

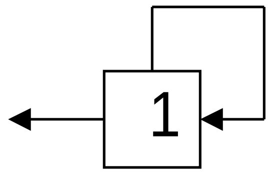
<u>Examples 1.</u>

a) $z_i = 0$, $i = 0,1,2,\ldots$ shortest LFSR:



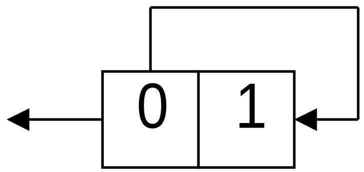
(no contents, length = 0)

b) $z_i = 1$, $i = 0,1,2,\ldots$ shortest LFSR:



(length $m = 1$)

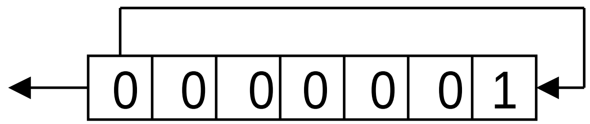
c) sequence 010101... ; shortest LFSR:



(length $m = 2$)

$z_0 = 0$, $z_1 = 1$, $z_{k+2} = z_k$, $k = 0,1,2,\ldots$

d) sequence 00000010000001 0... LFSR:



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The polynomial over $\mathbf{Z_2}$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_{m-1} x^{m-1} + x^m$$

is called the connection polynomial of the LFSR with taps $c_0\, c_1\, c_2 \ldots c_{m-1}$.

Given $f(x) = c_0 + c_1 x + \ldots + c_{m-1} x^{m-1} + x^m$ we denote by $f^*(x)$ the reciprocal polynomial of f, defined as follows:

$$f^*(x) = x^m f(x^{-1}) = c_0 x^m + c_1 x^{m-1} + c_2 x^{m-2} + \ldots + c_{m-1} x + 1.$$

It has the following properties:

1. $\deg f^*(x) \leq \deg f(x)$, and $\deg f^*(x) = \deg f(x)$ if and only if $c_0 = 1$.
2. Let $h(x) = f(x)g(x)$. Then $h^*(x) = f^*(x)g^*(x)$.

The set of sequences generated by the LFSR with connection polynomial $f(x)$ is denoted by $\Omega(f)$;

$$\Omega(f) = \{S = (z_i) \mid z_i \in \mathbf{Z_2};\ z_{k+m} = c_0 z_k + c_1 z_{k+1} + \ldots + c_{m-1} z_{k+m-1},\ k = 0,1,\ldots\}.$$

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$\Omega(f)$ is a linear space over $\mathbf{Z_2}$ of dimension m. Its elements S can also be expressed using the formal power series notation:

$$S = S(x) = z_0 + z_1 x + z_2 x^2 + z_3 x^3 + \ldots = \sum_{i=0\ldots\infty} z_i x^i$$

Theorem 1. If $S(x) \in \Omega(f)$, then there is a polynomial $P(x)$ of degree less than m (= deg $f(x)$) such that $S(x) = P(x)/f^*(x)$.

Proof. $f^*(x) = \sum_{i=0\ldots m} c_{m-i} x^i = \sum_{i=0\ldots\infty} c'_i x^i$, where $c_m = 1$, and $c'_i = c_{m-i}$, if $0 \leq i \leq m$, and $c'_i = 0$ otherwise. Then

$$S(x)f^*(x) = (\sum_{i=0\ldots\infty} z_i x^i)(\sum_{i=0\ldots\infty} c'_i x^i) = \sum_{i=0\ldots\infty} (\sum_{t=0\ldots i} z_{i-t} c'_t)\, x^i.$$

For $i \geq m$, denote $r = i - m$, and consider the i th term in the sum above:

$$\sum_{t=0\ldots i} z_{i-t} c'_t = \sum_{t=0\ldots m} z_{i-t} c'_t = \sum_{t=0\ldots m} z_{r+m-t} c_{m-t} = \sum_{k=0\ldots m} z_{r+k} c_k = 0,$$ if

$S(x) \in \Omega(f)$. Then $S(x)f^*(x) = \sum_{i=0\ldots m-1} (\sum_{t=0\ldots i} z_{i-t} c'_t)\, x^i = P(x)$.

Corollary 1. $\Omega(f) = \{ S(x) = P(x)/f^*(x) \mid \deg P(x) < \deg f(x) \}$.

Proof. Both sets are linear spaces over $\mathbf{Z_2}$ of the same dimension (deg $f(x)$). By Thm 1, $\Omega(f)$ is contained in the space on the right hand side. Therefore, the spaces are equal.

Theorem 2. Let $h(x) = \mathrm{lcm}\,(f(x), g(x))$, and let $S_1(x) \in \Omega(f)$ and $S_2(x) \in \Omega(g)$.

Then $S_1(x)+S_2(x) \in \Omega(h)$.

Proof. $h(x) = f(x)q_1(x) = g(x)q_2(x)$, where $\deg q_1(x) = \deg h(x) - \deg f(x)$ and $\deg q_2(x) = \deg h(x) - \deg g(x)$. Then by Thm 1:

$$S_1(x) + S_2(x) = (P_1(x)/f^*(x)) + (P_2(x)/g^*(x)) = (P_1(x)q_1^*(x) + P_2(x)q_2^*(x))/h^*(x)$$

where $\deg(P_1(x)q_1^*(x) + P_2(x)q_2^*(x)) \leq$

$$\max\{\deg P_1(x)+\deg q_1^*(x), \deg P_2(x)+ \deg q_2^*(x)\} < \deg h(x).$$

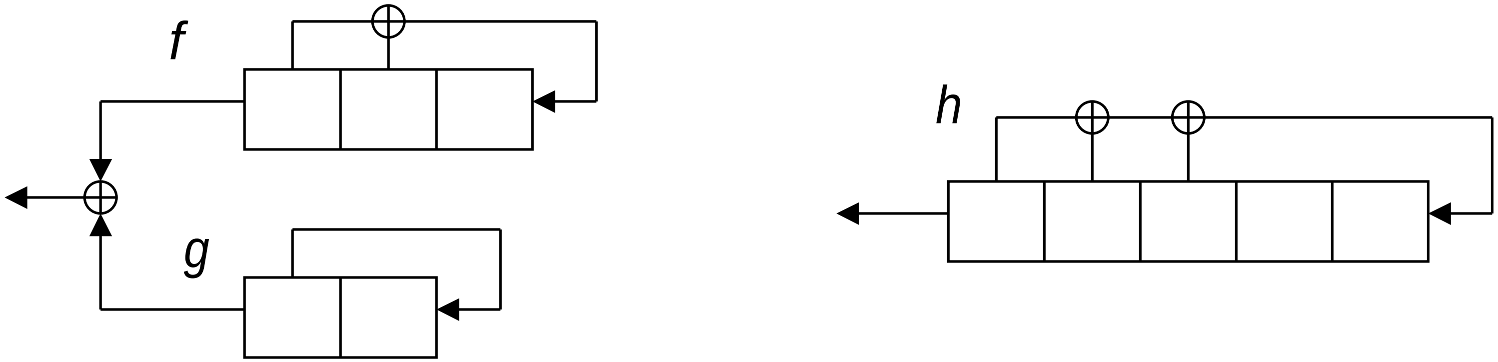
The claim follows using Corollary 1.

Corollary 2. If $f(x)$ divides $h(x)$, then $\Omega(f) \subset \Omega(h)$.

Example 2. $f(x) = x^3 + x + 1$; $g(x) = x^2 + 1$;

$h(x) = \text{lcm}\,(f(x),g(x)) = x^5 + x^2 + x + 1$.

All sequences generated by the LFSR combination on the left hand side can be generated using a single LFSR of length 5:



Further, if f-LFSR is initialized with 011, g-LFSR with 00, and the h-LFSR with 01110, then these two LFSRs generate the same sequence: 011100101110010…

Indeed, take the five first bits of any sequence generated by the f register and use them to initialize the h register. Then the h register generates the same sequence.

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In the example above the LFSR with connection polynomial $f(x)$ runs through all seven possible non-zero states.

The state space of the LFSR with polynomial $h(x)$ splits into five separate sets of states as follows:

00000	11111	01010	01110	10001	00001
		10101	11100	00011	00010
			11001	00110	00100
			10010	01101	01001
			00101	11010	10011
			01011	10100	00111
			10111	01000	01111
					11110
					11101
					11011
					10110
					01100
					11000
					10000

$$1 + 1 + 2 + 7 + 7 + 14 = 32 = 2^5$$

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FACT 1. For all binary polynomials $f(x)$ there is a polynomial of the form $x^e + 1$, where $e \geq 1$, such that $f(x)$ divides $x^e + 1$. The smallest of such non-negative integers e is called the exponent of $f(x)$. The exponent of $f(x)$ is divides all other numbers with this property.

If $S = (z_i) \in \Omega(x^e + 1)$, then clearly $z_i = z_{i+e}$, for all $i = 0,1,...$ Then it must be that the period of the sequence $S = (z_i)$ divides e.

We have the following theorem:

Theorem 3. If $S = (z_i) \in \Omega(f(x))$, then the period of S divides the exponent of $f(x)$.

FACT 2. There exist polynomials $f(x)$ for which all non-zero sequences in $\Omega(f)$ have a period equal to the exponent of $f(x)$. The polynomials with this property are exactly the irreducible polynomials.

FACT 3. For all positive integers m there exist polynomials of degree m with exponent equal to $2^m - 1$ (the largest possible value). Such polynomials are called primitive polynomials. Primitive polynomials are irreducible.

Corollary 3. Let $f(x)$ be a primitive polynomial of degree m. Then all sequences generated by an LFSR with polynomial $f(x)$ have period $2^m - 1$.

Example 4. Binary polynomials of degree 4 with non-zero constant term :

		exponent			exponent
$x^4 + 1 = (x + 1)^4$		4	$x^4 + x^2 + x + 1 = (x^3 + x^2 + 1)(x + 1)$		7
$x^4 + x + 1$	primitive	15	$x^4 + x^3 + x + 1 = (x + 1)^2(x^2 + x + 1)$		6
$x^4 + x^2 + 1 = (x^2 + x + 1)^2$		6	$x^4 + x^3 + x^2 + 1 = (x^3 + x + 1)(x + 1)$		7
$x^4 + x^3 + 1$	primitive	15	$x^4 + x^3 + x^2 + x + 1$	irreducible	5

LFSR 10/12 - Linear complexity

Let $S^{(m)} = z_0, z_1, z_2, z_3, \ldots, z_{m-1}$ be a finite sequence of length m. We say that the linear complexity LC($S^{(m)}$) of $S^{(m)}$ is the length of the shortest LFSR which generates the sequence $z_0, z_1, z_2, z_3, \ldots, z_{m-1}$.

Linear complexity does not decrease if new terms are added to the sequence, but it may remain the same.

Examples 5.

a) $S^{(m)}$ = 000...01 (with m - 1 zeroes); LC($S^{(m)}$) = m.

b) $S^{(m+1)}$ = 111..10 (with m ones); LC($S^{(m+1)}$) = m.

c) By example 3, the linear complexity of 0111001011 is less than or equal to 3. From b) it follows that the linear complexity is exactly 3.

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Theorem 4. Let $LC(S^{(m)}) = L$. Consider the LFSR of length L which generates the sequence $S^{(m)}$. Then

a) The L subsequent states of the the LFSR are linearly independent.

b) The $L + 1$ subsequent states are linearly dependent.

c) If moreover, at least $2L$ terms of the sequence are given, that is, $m \geq 2L$, then the connection polynomial of the generating LFSR is uniquely determined (cf. Stinson: Section 1.2.5).

Proof. Let the connection coefficients be $c_0\, c_1\, c_2\, c_3 \,...\, c_{L-1}$. Writing the recursion equation

$$z_{k+L} = c_0 z_k + c_1 z_{k+1} + c_2 z_{k+2} + ... + c_{L-1} z_{k+L-1}$$

in vector form we get

$$(c_0\ c_1\ c_2\ c_3\ ...\ c_{L-1})\, Z = (z_L\ z_{L+1}\ z_{L+2}\ z_{L+3}\ ...\ z_{2L-1}) \qquad (*)$$

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where the rows (and columns) of the matrix Z are vectors $(z_k\ z_{k+1}\ z_{k+2}\ z_{k+3}\ \ldots\ z_{k+L-1})$, for $k = 0,1,\ldots,L-1$. Claim b) follows immediately from this representation. Further, if L subsequent states are linearly dependent, the sequence satisfies a linear recursion relation of length (at most) $L-1$, and can be generated using a LFSR of length less than L. This gives a).

Finally, if at least $2L$ terms of the sequence are given, then the vectors

$$(z_k\ z_{k+1}\ z_{k+2}\ z_{k+3}\ \ldots\ z_{k+L-1}),\quad k = 0,1,\ldots,L$$

that determine the columns of the matrix Z in equation (*) are known. By a), the matrix Z is invertible. This gives a unique solution for the tap constants $(c_0\ c_1\ c_2\ c_3\ \ldots c_{L-1})$.