T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 11 ([NS, 1997], Predicate Logic, Chapters 10 – 14)
April 23–25, 2008

Solutions to demonstration problems

Solution to Problem 4

a) $U = \{c\}$, $B = \{G(c,c)\}$.

b) $U = \{a, f(a), f(f(a)), \ldots\}$, $B = \{P(e_1,e_2) \mid e_1 \in U, e_2 \in U\}$.

c) $U = \{a,b\}$, $B = \{P(a),P(b)\}$.

d) $U = \{a\}$, $B = \{P(a,a),G(a,a)\}$.

e) $U = \{a,b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$,
$B = \{P(e_1,e_2) \mid e_1 \in U, e_2 \in U\} \cup \{Q(e_1,e_2) \mid e_1 \in U, e_2 \in U\}$.

f) $U = \{a, f(a,a), f(a, f(a,a)), f(f(a,a),a), f(f(a,a), f(a,a)), \ldots\}$,
$B = \{P(e) \mid e \in U\} \cup \{Q(e) \mid e \in U\}$.

Solution to Problem 5

a) A clause $\{P(x,a,x)\}$ is obtained from the sentence $\forall x P(x,a,x)$. and the other sentence $\neg(\exists x \exists y \exists z (P(x,y,z) \wedge \neg P(x, f(y), f(z)))$ results in clause $\{\neg P(x,y,z), P(x, f(y), f(z))\}$. Thus we get

$$S = \{\{P(x,a,x)\}, \{\neg P(x,y,z), P(x, f(y), f(z))\}\}.$$

b) Herbrand-universe $H = \{a, f(a), f(f(a)), \ldots\} = \{f^n(a) \mid n \geq 0\}$ and Herbrand-base $B = \{P(e_1,e_2,e_3) \mid e_1,e_2,e_3 \in H\}$.

c) The maximal Herbrand-model for S is B, since every term of the form $P(f^n(a), a, f^n(a))$, $n \geq 0$ belongs to B (the first clause is satisfied), and each term of the form $P(f^n(a), f^{m+1}(a), f^{k+1}(a))$, for $n,m,k \geq 0$, belongs to B (the second clause is satisfied).

The minimal Herbrand-model is $\{P(a,a,a), P(a, f(a), f(a))\}$.

Solution to Problem 6

Find the set of clauses S which is the clausal form of the sentence (finite, contains no function symbols), find the Herbrand universe H of S and furthermore, the finite set of Herbrand-instances S'. This can be interpreted as a set of propositional clauses and for instance resolution can be used to check the validity of S'.

Solution to Problem 7

$$\{y/b, z/f(g(a)), w/c\}$$

Solution to Problem 8

a) $\sigma_0 = \varepsilon$ (empty substitution)
$S_0 = \{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_0) = \{x, f(y)\}$
$\sigma_1 = \{x/f(y)\}$
$\sigma_0\sigma_1 = \{x/f(y)\}$
$S_1 = \{P(f(y), g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_1) = \{y, f(z)\}$
$\sigma_2 = \{y/f(z)\}$
$\sigma_0\sigma_1\sigma_2 = \{x/f(f(z)), y/f(z)\}$
$S_2 = \{P(f(f(z)), g(f(z)), f(a)), P(f(f(z)), g(f(z)), z)\}$
$D(S_2) = \{f(a), z\}$
$\sigma_3 = \{z/f(a)\}$
$\sigma_0\sigma_1\sigma_2\sigma_3 = \{x/f(f(f(a))), y/f(f(a)), z/f(a)\}$
$S_3 = \{P(f(f(f(a))), g(f(f(a))), f(a))\}$
MGU is $\sigma_0\sigma_1\sigma_2\sigma_3$.

b) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_0) = \{x, a, y\}$
$\sigma_1 = \{x/a\}$
$S_1 = \{P(a, f(a), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_1) = \{a, y\}$
$\sigma_2 = \{y/a\}$
$S_2 = \{P(a, f(a), g(a)), P(a, f(g(a)), g(a))\}$
$D(S_2) = \{a, g(a)\}$
Terms a and $g(a)$ cannot be unified.

c) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x, y)), P(y, f(y, a)), P(b, f(b, a))\}$
$D(S_0) = \{x, y, b\}$

$\sigma_1 = \{x/b\}$
$S_1 = \{P(b, f(b,y)), P(y, f(y,a)), P(b, f(b,a))\}$
$D(S_1) = \{b, y\}$
$\sigma_2 = \{y/b\}$
$S_2 = \{P(b, f(b,b)), P(b, f(b,a))\}$
$D(S_2) = \{b, a\}$
Terms b and a cannot be unified.

d) $\sigma_0 = \varepsilon$
$S_0 = \{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$
$D(S_0) = \{f(a), y, x\}$
$\sigma_1 = \{y/f(a)\}$
$S_1 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(x, f(a), f(z))\}$
$D(S_1) = \{f(a), x\}$
$\sigma_2 = \{x/f(a)\}$
$S_2 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(f(a), f(a), f(z))\}$
$D(S_2) = \{z, b, f(z)\}$
$\sigma_3 = \{z/b\}$
$S_3 = \{P(f(a), f(a), b), P(f(a), f(a), f(b))\}$
$D(S_3) = \{b, f(b)\}$
Terms b and $f(b)$ cannot be unified.

Solution to Problem 9

a) Consider $\sigma = \{x/a\}$ and $\lambda = \{x/b\}$. Now, $\sigma\lambda \neq \lambda\sigma$.

b) $S = \{P(x), P(y)\}$ has two MGUs: $\{x/y\}$ and $\{y/x\}$.

Solution to Problem 10

$$\{x/f(w,w), y/f(f(w,w), f(w,w)),$$
$$z/f(f(f(w,w), f(w,w)), f(f(w,w), f(w,w)))\}.$$

Solution to Problem 11

Define $P(x)$ = "x is barber" and $A(x,y)$ = "x shaves y".

a) $\forall x(P(x) \rightarrow \forall y(\neg A(y,y) \rightarrow A(x,y)))$,

b) $\forall x(P(x) \rightarrow \forall y(A(y,y) \rightarrow \neg A(x,y)))$.

The clausal form:

a) $\forall x(P(x) \to \forall y(\neg A(y,y) \to A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(A(y,y) \vee A(x,y)))$
$\forall x \forall y(\neg P(x) \vee A(y,y) \vee A(x,y))$
$\neg P(x) \vee A(y,y) \vee A(x,y)$
$\{\neg P(x_1),\ A(y_1,y_1),\ A(x_1,y_1)\}$

b) $\forall x(P(x) \to \forall y(A(y,y) \to \neg A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(\neg A(y,y) \vee \neg A(x,y)))$
$\forall x \forall y(\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y))$
$\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y)$
$\{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$

We want to show $\neg \exists x P(x)$, and thus consider its negation $\exists x P(x)$. In the clausal form: $\{P(a)\}$.
From clauses

$$\{\neg P(x_1),\ A(y_1,y_1),\ A(x_1,y_1)\} \quad \text{and} \quad \{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$$

we get

$$\{\neg P(x_3)\} \qquad (\text{substitution } \{x_1/x_3, x_2/x_3, y_1/x_3, y_2/x_3\})$$

From clauses $\{P(a)\}$ and $\{\neg P(x_3)\}$ we obtain the empty clause (substitution $\{x_3/a\}$). Thus the set of clauses is unsatisfiable and $\neg \exists x P(x)$ is a logical consequence of the premises.

Solution to Problem 12
We start with the base cases, that is, 0 is divisible by two and three:

$$J2(0),$$
$$J3(0).$$

Furthermore, divisibility for larger numbers:

$$\forall x(J2(x) \to J2(s(s(x)))),$$
$$\forall x(J3(x) \to J3(s(s(s(x))))).$$

Finally, divisibility by six:

$$\forall x(J2(x) \wedge J3(x) \to J6(x)).$$

We transform the sentences into clausal form. For the definition of predicate $J2(x)$ we get:

$$\begin{aligned}
&\forall x(J2(x) \rightarrow J2(s(s(x))))\\
&\forall x(\neg J2(x) \vee J2(s(s(x)))\\
&\{\neg J2(x), J2(s(s(x)))\}.
\end{aligned}$$

Similarly for the definition of predicate $J3(x)$ we obtain $\{\neg J3(x), J3(s(s(s(x))))\}$. The sentence defining predicate $J6(x)$ results in the following:

$$\begin{aligned}
&\forall x(J2(x) \wedge J3(x) \rightarrow J6(x))\\
&\forall x(\neg(J2(x) \wedge J3(x)) \vee J6(x))\\
&\forall x(\neg J2(x) \vee \neg J3(x) \vee J6(x))\\
&\{\neg J2(x), \neg J3(x), J6(x)\}.
\end{aligned}$$

From the negation of the query we obtain the following three clauses:

$$\begin{aligned}
&\neg\forall x(J2(x) \wedge J3(x) \rightarrow J6(s^6(x)))\\
&\neg\forall x(\neg(J2(x) \wedge J3(x)) \vee J6(s^6(x)))\\
&\neg\forall x(\neg J2(x) \vee \neg J3(x)) \vee J6(s^6(x)))\\
&\exists x\neg(\neg J2(x) \vee \neg J3(x) \vee J6(s^6(x)))\\
&\exists x(J2(x) \wedge J3(x) \wedge \neg J6(s^6(x)))\\
&\{J2(c)\}, \{J3(c)\} \text{ and } \{\neg J6(s^6(c))\}.
\end{aligned}$$

The resolution refutation:

1. $\{J2(c)\}$, P
2. $\{\neg J2(x_1), J2(s(s(x_1)))\}$, P
3. $\{J2(s(s(c)))\}$, 1 & 2, x_1/c
4. $\{\neg J2(x_2), J2(s(s(x_2)))\}$, P
5. $\{J2(s^4(c))\}$, 3 & 4, $x_2/s(s(c))$
6. $\{\neg J2(x_3), J2(s(s(x_3)))\}$, P
7. $\{J2(s^6(c))\}$, 5 & 6, $x_3/s^6(c)$
8. $\{J3(c)\}$, P
9. $\{\neg J3(x_4), J3(s(s(s(x_4))))\}$, P
10. $\{J3(s(s(s(c))))\}$, 8 & 9, x_4/c
11. $\{\neg J3(x_5), J3(s(s(s(x_5))))\}$, P
12. $\{J3(s^6(c))\}$, 10 & 11, $x_4/s(s(s(c)))$
13. $\{\neg J2(x_6), \neg J3(x_6), J6(x_6)\}$, P
14. $\{\neg J3(s^6(c)), J6(s^6(c))\}$, 7 & 13, $x_6/s^6(c)$
15. $\{J6(s^6(c))\}$, 12 & 14
16. $\{\neg J6(s^6(c))\}$, P
17. $\square$, 15 & 16