T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 8 ([NS, 1997], Predicate Logic, Chapters 4 and 9)
March 27–29, 2007

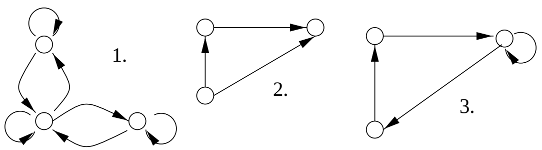
Solutions to demonstration problems

4. Let R be a binary predicate with interpretation $R^{\mathcal{S}} \subseteq U \times U$ (the set U is the domain of structure $\mathcal{S}$). In the following table we give definitions for some properties of relation $R^{\mathcal{S}}$.

Property	Definition
reflexivity	$\forall x R(x,x)$
irreflexivity	$\forall x \neg R(x,x)$
symmetry	$\forall x \forall y (R(x,y) \rightarrow R(y,x))$
asymmetry	$\forall x \forall y (R(x,y) \rightarrow \neg R(y,x))$
transitivity	$\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$
seriality	$\forall x \exists y R(x,y)$

Consider a domain U consisting of people. Give examples of relations $R^{\mathcal{S}}$, ($\emptyset \subset R^{\mathcal{S}} \subset U^2$), that have properties described above.

Solution. The graphs given below illustrate different properties of relations. Here the nodes are the elements in a structure and there is an edge between two nodes $x \in A, y \in A$ if and only if $R(x,y)$ is true for x,y.



Reflexivity ($\forall x R(x,x)$) means that every node in the graph has an edge to itself and irreflexivity ($\forall x \neg R(x,x)$) means that no node has an edge to itself. First of the graphs is reflexive, the second irreflexive and the third is neither reflexive nor irreflexive.

Symmetry ($\forall x \forall y (R(x,y) \rightarrow R(y,x))$) means that whenever there is an edge from x to y, there is also an edge from y to x. Asymmetric ($\forall x \forall y (R(x,y) \rightarrow \neg R(y,x))$) graph has no edge from y to x if there is edge from x to y. The first graph is symmetric, the second asymmetric and the third is neither.

In a transitive graph ($\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$) if there is a path from x to y along the edges, then there is an edge from x to y in the graph. The second graph is transitive.

In a serial graph ($\forall x \exists y R(x,y)$) there is at least one edge from each node x. The first and the third graph are serial.

Now define relations $T(x,y)$ (x knows y), $N(x,y)$ (x is married to y), $V(x,y)$ (y is a parent of x) ja $E(x,y)$ (y is an ancestor of x). There relations have the following properties.

Relation	refl.	irrefl.	symm.	asymm.	trans.	serial.
knows	*		*			*
married to		*	*			
parent		*		*		*
ancestor		*		*	*	*

5. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.

a) $\forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y)$

b) $\exists x (P(x) \vee Q(x)) \rightarrow \exists x P(x) \wedge \exists x Q(x)$

c) $\neg \forall x (P(x) \rightarrow R(x)) \vee \neg \forall x (P(x) \rightarrow \neg R(x))$

Solution.

a) Consider $\mathcal{S}$ with domain $U = \{1,2\}$ and $P^{\mathcal{S}} = \{\langle 1,1 \rangle, \langle 2,2 \rangle\}$. Now it holds $\mathcal{S} \models \forall x \exists y P(x,y)$ and $\mathcal{S} \not\models \exists y \forall x P(x,y)$ (there is no value for y such that for all x we would have $\langle x,y \rangle \in P^{\mathcal{S}}$). Thus the implication is false in $\mathcal{S}$.

b) Consider $\mathcal{S}$ with domain $U = \{1\}$ and $P^{\mathcal{S}} = \{1\}, Q^{\mathcal{A}} = \emptyset$. Now the left side of the implication is true and the right side false in $\mathcal{S}$, and $\mathcal{S}$ is a counterexample.

c) Consider $\mathcal{S}$ with domain $U = \{1\}$ ja $P^{\mathcal{S}} = \emptyset, R^{\mathcal{S}} = \{1\}$. Now $\forall x (P(x) \rightarrow R(x))$ is true in $\mathcal{S}$ since the left side of the implication is false in $\mathcal{S}$. Similarly $\mathcal{S} \models \forall x (P(x) \rightarrow \neg R(x))$.

6. Transform the following sentences into conjunctive normal form and perform skolemization.

a) $\forall y (\exists x P(x,y) \rightarrow \forall z Q(y,z)) \wedge \exists y (\forall x R(x,y) \vee \forall x Q(x,y))$

b) $\exists x \forall y R(x,y) \leftrightarrow \forall y \exists x P(x,y)$

c) $\forall x \exists y Q(x,y) \vee (\exists x \forall y P(x,y) \wedge \neg \exists x \exists y P(x,y))$

d) $\neg (\forall x \exists y P(x,y) \rightarrow \exists x \exists y R(x,y)) \wedge \forall x \neg \exists y Q(x,y)$

Solution.

- Remove connectives $\to$ and $\leftrightarrow$.
- Negations in, quantifiers out.
- Use distribution rules to obtain CNF / DNF.

a)

$$\begin{aligned}
&\forall y(\exists xP(x,y) \to \forall zQ(y,z)) \wedge \exists y(\forall xR(x,y) \vee \forall xQ(x,y))\\
\equiv&\forall y(\neg\exists xP(x,y) \vee \forall zQ(y,z)) \wedge \exists y(\forall xR(x,y) \vee \forall xQ(x,y))\\
\equiv&\forall y(\forall x\neg P(x,y) \vee \forall zQ(y,z)) \wedge \exists y(\forall xR(x,y) \vee \forall xQ(x,y))\\
\equiv&\exists y_1(\forall y(\forall x\neg P(x,y) \vee \forall zQ(y,z)) \wedge (\forall xR(x,y_1) \vee \forall xQ(x,y_1)))\\
\equiv&\exists y_1\forall y_2((\forall x\neg P(x,y_2) \vee \forall zQ(y_2,z)) \wedge (\forall xR(x,y_1) \vee \forall xQ(x,y_1)))\\
\equiv&\exists y_1\forall y_2\forall x_1\forall x_2\forall z\forall x_3((\neg P(x_1,y_2) \vee Q(y_2,z)) \wedge (R(x_2,y_1) \vee Q(x_3,y_1)))
\end{aligned}$$

This is the Prenex normal form and the part inside quantifiers is in CNF. Skolemization:

$$\forall y_2\forall x_1\forall x_2\forall z\forall x_3((\neg P(x_1,y_2) \vee Q(y_2,z)) \wedge (R(x_2,c) \vee Q(x_3,c)))$$

c)

$$\begin{aligned}
&\forall x\exists yQ(x,y) \vee (\exists x\forall yP(x,y) \wedge \neg\exists x\exists yP(x,y))\\
\equiv&\forall x\exists yQ(x,y) \vee (\exists x\forall yP(x,y) \wedge \forall x\forall y\neg P(x,y))\\
\equiv&\forall x\exists yQ(x,y) \vee \exists x_1\forall y_1\forall x_2\forall y_2(P(x_1,y_1) \wedge \neg P(x_2,y_2))\\
\equiv&\exists x_1\forall x_3\exists y_3\forall y_1\forall x_2\forall y_2(Q(x_3,y_3) \vee (P(x_1,y_1) \wedge \neg P(x_2,y_2)))
\end{aligned}$$

This is the Prenex normal form and we continue to get CNF.

$$\exists x_1\forall x_3\exists y_3\forall y_1\forall x_2\forall y_2((Q(x_3,y_3) \vee P(x_1,y_1)) \wedge (Q(x_3,y_3) \vee \neg P(x_2,y_2)))$$

Skolemization:

$$\forall x_3\forall y_1\forall x_2\forall y_2((Q(x_3,f(x_3)) \vee P(c,y_1)) \wedge (Q(x_3,f(x_3)) \vee \neg P(x_2,y_2)))$$

7. Use the rules in Lemma 9.1 [NS, 1997, page 129] to obtain rules for the following cases.

a) $\forall x\phi(x) \to \psi$

b) $\exists x\phi(x) \to \psi$

c) $\phi \to \forall x\psi(x)$

d) $\phi \to \exists x\psi(x)$

Solution.

a)

$$\begin{aligned}
&\forall x\phi(x) \to \psi\\
\equiv&\neg\forall x\phi(x) \vee \psi\\
\equiv&\exists x\neg\phi(x) \vee \psi\\
\equiv&\exists x_1(\neg\phi(x_1) \vee \psi)\\
\equiv&\exists x_1(\phi(x_1) \to \psi)
\end{aligned}$$

b) Similarly, $\exists x\phi(x) \to \psi \equiv \forall x_1(\phi(x_1) \to \psi)$.

c)

$$\begin{aligned}
&\phi \to \forall x\psi(x)\\
\equiv&\neg\phi \vee \forall x\psi(x)\\
\equiv&\forall x_1(\neg\phi \vee \psi(x_1))\\
\equiv&\forall x_1(\phi \to \psi(x_1))
\end{aligned}$$

d) Similarly, $\phi \to \exists x\psi(x) \equiv \exists x_1(\phi \to \psi(x_1))$.

8. Transform the following sentences into clausal form.

a) $\neg\exists x((P(x) \to P(a)) \wedge (P(x) \to P(b)))$

b) $\forall y\exists xP(x,y)$

c) $\neg\forall y\exists xG(x,y)$

d) $\exists x\forall y\exists z(P(x,z) \vee P(z,y) \to G(x,y))$

Solution.

a) Sentence $\neg\exists x((P(x) \to P(a)) \wedge (P(x) \to P(b)))$:
Eliminate implications: $\neg\exists x((\neg P(x) \vee P(a)) \wedge (\neg P(x) \vee P(b)))$.
Push $\neg$ inside $\exists x$:
$\forall x\neg((\neg P(x) \vee P(a)) \wedge (\neg P(x) \vee P(b)))$.

Push negations inside the formula:

$\forall x((P(x) \land \neg P(a)) \lor (P(x) \land \neg P(b)))$.

Bring $P(x)$ outside: $\forall x(P(x) \land (\neg P(a) \lor \neg P(b)))$.

Drop universal quantifiers: $P(x) \land (\neg P(a) \lor \neg P(b))$.

Clausal form: $\{\{P(x)\},\ \{\neg P(a),\ \neg P(b)\}\}$.

b) Sentence $\forall y \exists x P(x,y)$:

Skolemization: $\forall y P(f(y),y)$.

Drop universal quantifiers: $P(f(y),y)$.

Clausal form: $\{\{P(f(y),y)\}\}$.

c) Sentence $\neg \forall y \exists x G(x,y)$:

Push $\neg$ inside $\forall y$: $\exists y \neg \exists x G(x,y)$.

Push $\neg$ inside $\exists x$: $\exists y \forall x \neg G(x,y)$

Skolemization: $\forall x \neg G(x,c)$.

Drop universal quantifiers: $\neg G(x,c)$.

Clausal form: $\{\{\neg G(x,c)\}\}$.

d) Sentence $\exists x \forall y \exists z (P(x,z) \lor P(z,y) \rightarrow G(x,y))$:

Eliminate implication: $\exists x \forall y \exists z (\neg (P(x,z) \lor P(z,y)) \lor G(x,y))$.

Push negations inside:

$\exists x \forall y \exists z ((\neg P(x,z) \land \neg P(z,y)) \lor G(x,y))$.

Push $G(x,y)$ inside the formula:

$\exists x \forall y \exists z ((\neg P(x,z) \lor G(x,y)) \land (\neg P(z,y) \lor G(x,y)))$.

Skolemization: $\forall y \exists z ((\neg P(c,z) \lor G(c,y)) \land (\neg P(z,y) \lor G(c,y)))$.

Skolemization: $\forall y ((\neg P(c,f(y)) \lor G(c,y)) \land (\neg P(f(y),y) \lor G(c,y)))$.

Drop universal quantifiers:

$(\neg P(c,f(y)) \lor G(c,y)) \land (\neg P(f(y),y) \lor G(c,y))$.

Clausal form:

$\{\{\neg P(c,f(y)), G(c,y)\},\ \{\neg P(f(y),y), G(c,y)\}\}$.