T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 11 ([NS, 1997], Predicate Logic, Chapters 10 – 14)
April 24–26, 2007

Solutions to demonstration problems

4. Define the Herbrand universe and Herbrand base for the following sets of clauses.

a) $\{\{\neg G(x,c)\}\}$,

b) $\{\{P(f(y),y)\}\}$,

c) $\{\{P(x)\},\ \{\neg P(a), \neg P(b)\}\}$,

d) $\{\{\neg P(x,y), \neg P(y,z), G(x,z)\}\}$,

e) $\{\{\neg P(x,y)\},\ \{Q(a,x), Q(b, f(y))\}\}$, ja

f) $\{\{P(x), Q(f(x,y))\}\}$

Solution.

a) $U = \{c\}, B = \{G(c,c)\}$.

b) $U = \{a, f(a), f(f(a)), \ldots\}, B = \{P(e_1,e_2) \,|\, e_1 \in U, e_2 \in U\}$.

c) $U = \{a,b\}, B = \{P(a), P(b)\}$.

d) $U = \{a\}, B = \{P(a,a), G(a,a)\}$.

e) $U = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$,
$B = \{P(e_1,e_2) \,|\, e_1 \in U, e_2 \in U\} \cup \{Q(e_1,e_2) \,|\, e_1 \in U, e_2 \in U\}$.

f) $U = \{a, f(a,a), f(a, f(a,a)), f(f(a,a),a), f(f(a,a), f(a,a)), \ldots\}$,
$B = \{P(e) \,|\, e \in U\} \cup \{Q(e) \,|\, e \in U\}$.

5. Consider

$$\Sigma = \{\forall x P(x,a,x),\ \neg\exists x \exists y \exists z (P(x,y,z) \wedge \neg P(x, f(y), f(z)))\}.$$

a) Transform Σ into a set of clauses S.

b) Define the Herbrand universe H and Herbrand base B of S.

c) Let Herbrand structures be subsets of the Herbrand base. Find the subset minimal and maximal Herbrand models of S.

Solution.

a) A clause $\{P(x,a,x)\}$ is obtained from the sentence $\forall x P(x,a,x)$. and the other sentence $\neg(\exists x \exists y \exists z (P(x,y,z) \wedge \neg P(x, f(y), f(z)))$ results in clause $\{\neg P(x,y,z), P(x, f(y), f(z))\}$. Thus we get

$$S = \{\{P(x,a,x)\}, \{\neg P(x,y,z), P(x, f(y), f(z))\}\}.$$

b) Herbrand-universe $H = \{a, f(a), f(f(a)), \ldots\} = \{f^n(a) \mid n \geq 0\}$ and Herbrand-base $B = \{P(e_1,e_2,e_3) \mid e_1,e_2,e_3 \in H\}$.

c) The maximal Herbrand-model for S is B, since every term of the form $P(f^n(a), a, f^n(a))$, $n \geq 0$ belongs to B (the first clause is satisfied), and each term of the form $P(f^n(a), f^{m+1}(a), f^{k+1}(a))$, for $n,m,k \geq 0$, belongs to B (the second clause is satisfied).
The minimal Herbrand-model is $\{P(a,a,a), P(a, f(a), f(a))\}$.

6. Transform the problem of deciding the validity of sentence

$$\exists x \exists y (P(x) \leftrightarrow \neg P(y)) \rightarrow \exists x \exists y (\neg P(x) \wedge P(y))$$

into the problem of satisfiability of a propositional logic statement and solve the problem.

Solution. Find the set of clauses S which is the clausal form of the sentence (finite, contains no function symbols), find the Herbrand universe H of S and furthermore, the finite set of Herbrand-instances S'. This can be interpreted as a set of propositional clauses and for instance resolution can be used to check the validity of S'.

7. Find the composition of substitutions $\{x/y, y/b, z/f(x)\}$ and $\{x/g(a), y/x, w/c\}$.

Solution.

$$\{y/b, z/f(g(a)), w/c\}$$

8. Find the most general unifiers for the following sets of literals.

a) $\{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$

b) $\{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$

c) $\{P(x, f(x,y)), P(y, f(y,a)), P(b, f(b,a))\}$

d) $\{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$

Solution.

a) $\sigma_0 = \varepsilon$ (empty substitution)
$S_0 = \{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_0) = \{x, f(y)\}$
$\sigma_1 = \{x/f(y)\}$
$\sigma_0\sigma_1 = \{x/f(y)\}$
$S_1 = \{P(f(y), g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_1) = \{y, f(z)\}$
$\sigma_2 = \{y/f(z)\}$
$\sigma_0\sigma_1\sigma_2 = \{x/f(f(z)), y/f(z)\}$
$S_2 = \{P(f(f(z)), g(f(z)), f(a)), P(f(f(z)), g(f(z)), z)\}$
$D(S_2) = \{f(a), z\}$
$\sigma_3 = \{z/f(a)\}$
$\sigma_0\sigma_1\sigma_2\sigma_3 = \{x/f(f(f(a))), y/f(f(a)), z/f(a)\}$
$S_3 = \{P(f(f(f(a))), g(f(f(a))), f(a))\}$
MGU is $\sigma_0\sigma_1\sigma_2\sigma_3$.

b) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_0) = \{x, a, y\}$
$\sigma_1 = \{x/a\}$
$S_1 = \{P(a, f(a), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_1) = \{a, y\}$
$\sigma_2 = \{y/a\}$
$S_2 = \{P(a, f(a), g(a)), P(a, f(g(a)), g(a))\}$
$D(S_2) = \{a, g(a)\}$
Terms a and $g(a)$ cannot be unified.

c) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x, y)), P(y, f(y, a)), P(b, f(b, a))\}$
$D(S_0) = \{x, y, b\}$
$\sigma_1 = \{x/b\}$
$S_1 = \{P(b, f(b, y)), P(y, f(y, a)), P(b, f(b, a))\}$
$D(S_1) = \{b, y\}$
$\sigma_2 = \{y/b\}$
$S_2 = \{P(b, f(b, b)), P(b, f(b, a))\}$
$D(S_2) = \{b, a\}$
Terms b and a cannot be unified.

d) $\sigma_0 = \varepsilon$
$S_0 = \{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$
$D(S_0) = \{f(a), y, x\}$
$\sigma_1 = \{y/f(a)\}$
$S_1 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(x, f(a), f(z))\}$
$D(S_1) = \{f(a), x\}$
$\sigma_2 = \{x/f(a)\}$
$S_2 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(f(a), f(a), f(z))\}$
$D(S_2) = \{z, b, f(z)\}$
$\sigma_3 = \{z/b\}$
$S_3 = \{P(f(a), f(a), b), P(f(a), f(a), f(b))\}$
$D(S_3) = \{b, f(b)\}$
Terms b and $f(b)$ cannot be unified.

9. Show that

a) the composition of substitutions is not commutative, that is, there are substitutions σ and λ such that $\sigma\lambda \neq \lambda\sigma$.

b) a most general unifier is not unique, that is, there is a set of literals S such that it has two most general unifiers σ and λ such that $\sigma \neq \lambda$.

Solution.

a) Consider $\sigma = \{x/a\}$ and $\lambda = \{x/b\}$. Now, $\sigma\lambda \neq \lambda\sigma$.

b) $S = \{P(x), P(y)\}$ has two MGUs: $\{x/y\}$ and $\{y/x\}$.

10. Unify $\{P(x, y, z), P(f(w, w), f(x, x), f(y, y))\}$.

Solution.

$$\{x/f(w,w), y/f(f(w,w), f(w,w)),$$
$$z/f(f(f(w,w), f(w,w)), f(f(w,w), f(w,w)))\}.$$

11. Use resolution to prove that there are no barbers, when

a) all barbers shave everyone who does not shave himself, and

b) no barber shaves anyone who shaves himself.

Solution. Define $P(x)$ = "x is barber" and $A(x, y)$ = "x shaves y".

a) $\forall x(P(x) \rightarrow \forall y(\neg A(y, y) \rightarrow A(x, y)))$,

b) $\forall x(P(x) \rightarrow \forall y(A(y, y) \rightarrow \neg A(x, y)))$.

The clausal form:

a) $\forall x(P(x) \rightarrow \forall y(\neg A(y,y) \rightarrow A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(A(y,y) \vee A(x,y)))$
$\forall x \forall y(\neg P(x) \vee A(y,y) \vee A(x,y))$
$\neg P(x) \vee A(y,y) \vee A(x,y)$
$\{\neg P(x_1),\ A(y_1,y_1),\ A(x_1,y_1)\}$

b) $\forall x(P(x) \rightarrow \forall y(A(y,y) \rightarrow \neg A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(\neg A(y,y) \vee \neg A(x,y)))$
$\forall x \forall y(\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y))$
$\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y)$
$\{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$

We want to show $\neg \exists x P(x)$, and thus consider its negation $\exists x P(x)$. In the clausal form: $\{P(a)\}$.

From clauses

$\{\neg P(x_1),\ A(y_1,y_1),\ A(x_1,y_1)\}$ and $\{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$

we get

$\{\neg P(x_3)\}$ (substitution $\{x_1/x_3, x_2/x_3, y_1/x_3, y_2/x_3\}$)

From clauses $\{P(a)\}$ and $\{\neg P(x_3)\}$ we obtain the empty clause (substitution $\{x_3/a\}$). Thus the set of clauses is unsatisfiable and $\neg \exists x P(x)$ is a logical consequence of the premises.

12. We use groud terms 0, $s(0)$, $s(s(0))$, $\ldots$, to represent natural numbers 0, 1, 2, $\ldots$, where 0 is a constants and s is a unary function such that $s(x) = x+1$ for all natural numbers x.

a) Let predicates $J2(x), J3(x)$ and $J6(x)$ represent that a natural number x is divisible by two, three and six, respectively. Define these predicates with sentences in predicate logic using the definitions of $J2$ and $J3$ to define $J6$.

b) Use resolution to prove that if a natural number x is divisible by two and three, then natural number $x+6$ is divisible by six.

Solution. We start with the base cases, that is, 0 is divisible by two and three:

$$J2(0),$$
$$J3(0).$$

Furthermore, divisibility for larger numbers:

$$\forall x(J2(x) \rightarrow J2(s(s(x)))),$$
$$\forall x(J3(x) \rightarrow J3(s(s(s(x))))).$$

Finally, divisibility by six:

$$\forall x(J2(x) \wedge J3(x) \rightarrow J6(x)).$$

We transform the sentences into clausal form. For the definition of predicate $J2(x)$ we get:

$$\forall x(J2(x) \rightarrow J2(s(s(x))))$$
$$\forall x(\neg J2(x) \vee J2(s(s(x)))$$
$$\{\neg J2(x), J2(s(s(x)))\}.$$

Similarly for the definition of predicate $J3(x)$ we obtain $\{\neg J3(x), J3(s(s(s(x))))\}$. The sentence defining predicate $J6(x)$ results in the following:

$$\forall x(J2(x) \wedge J3(x) \rightarrow J6(x))$$
$$\forall x(\neg(J2(x) \wedge J3(x)) \vee J6(x))$$
$$\forall x(\neg J2(x) \vee \neg J3(x) \vee J6(x))$$
$$\{\neg J2(x), \neg J3(x), J6(x)\}.$$

From the negation of the query we obtain the following three clauses:

$$\neg \forall x(J2(x) \wedge J3(x) \rightarrow J6(s^6(x)))$$
$$\neg \forall x(\neg(J2(x) \wedge J3(x)) \vee J6(s^6(x)))$$
$$\neg \forall x(\neg J2(x) \vee \neg J3(x)) \vee J6(s^6(x)))$$
$$\exists x \neg(\neg J2(x) \vee \neg J3(x) \vee J6(s^6(x)))$$
$$\exists x(J2(x) \wedge J3(x) \wedge \neg J6(s^6(x)))$$
$$\{J2(c)\}, \{J3(c)\} \text{ and } \{\neg J6(s^6(c))\}.$$

The resolution refutation:

1. $\{J2(c)\}$, P
2. $\{\neg J2(x_1), J2(s(s(x_1)))\}$, P
3. $\{J2(s(s(c)))\}$, 1 & 2, x_1/c
4. $\{\neg J2(x_2), J2(s(s(x_2)))\}$, P
5. $\{J2(s^4(c))\}$, 3 & 4, $x_2/s(s(c))$
6. $\{\neg J2(x_3), J2(s(s(x_3)))\}$, P
7. $\{J2(s^6(c))\}$, 5 & 6, $x_3/s^6(c)$
8. $\{J3(c)\}$, P
9. $\{\neg J3(x_4), J3(s(s(s(x_4))))\}$, P
10. $\{J3(s(s(s(c))))\}$, 8 & 9, x_4/c
11. $\{\neg J3(x_5), J3(s(s(s(x_5))))\}$, P
12. $\{J3(s^6(c))\}$, 10 & 11, $x_4/s(s(s(c)))$
13. $\{\neg J2(x_6), \neg J3(x_6), J6(x_6)\}$, P
14. $\{\neg J3(s^6(c)), J6(s^6(c))\}$, 7 & 13, $x_6/s^6(c)$
15. $\{J6(s^6(c))\}$, 12 & 14
16. $\{\neg J6(s^6(c))\}$, P
17. $\square$, 15 & 16