Rapid Mixing via Conductance

Evimaria Terzi

References

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Classes of Problems

Basic Notations

- x: Problem Instance.
- $R(x)$: Finite set of structures (solutions to instance x).
- Fix a finite alphabet $\Sigma \supseteq \{0, 1\}$ over which instances and solutions are encoded. Then

$$R \subseteq \Sigma^* \times \Sigma^*$$

and $\forall x \in \Sigma^*$ the corresponding solution set wrt R is just

$$R(x) = \{y \in \Sigma^* : \langle x, y \rangle \in R\}$$

Classes of Problems

Example

- *Problem Instance:* Boolean Formula B in Disjunctive Normal Form (DNF)
- *Solution Set* $R(B)$: All satisfactory assignments of B
- *Formally:*

$$\begin{aligned} R = \{\langle x, y\rangle : x \in \Sigma^* &\text{ encodes boolean formula } B \text{ in } DNF \\ y \in \Sigma^* &\text{ encodes satisfying assignment of } B\} \end{aligned}$$

Basic Terminology and Notation

- $(X_t)_{t=0}^{\infty}$: time homogeneous Markov chain.
- $[N] = \{0, 1, ..., N\}$ with $N \geq 1$ is the finite state space of the chain.
- $P = (p_{ij})_{i,j=0}^{N-1}$: transition matrix of the Markov chain (non-negative stochastic matrix).
- $P_{ij} = Pr(X_{t+1} = j | X_t = i)$ is the transition probability from state i to state j.
- $P^s = (p_{ij}^s)$ with $s \in \mathbb{N}$ and $p_{ij}^s = Pr(X_{t+s} = j | X_t = i)$ independent of t.

Ergodicity

- Let $\pi^{(t)} = (\pi_i^t)_{i=0}^{N-1}$ so that $\pi_i^{(t)} = Pr(X_t = i)$ then $\pi^{(0)}$ denotes the *initial distribution* and $\pi^{(t)} = \pi^{(0)} P^t$ for all $t \in \mathbb{N}$.
- The chain is *ergodic* if there exists a distribution $\pi = (\pi_i) > 0$ over $[N]$ such that
$$\lim_{s \to \infty} p_{ij}^s = \pi_j \; \forall i, j \in [N]$$
In this case, we have that $\pi^{(t)} = \pi^{(0)} P^t \to \pi$ pointwise as $t \to \infty$, and the limit is independent of $\pi^{(0)}$.
- The *stationary distribution* π is the unique vector satisfying $\pi P = \pi$, with $\sum_i \pi_i = 1$ i.e. is the unique normalized left eigenvector of P with eigenvalue 1.
- Necessary and sufficient conditions for ergodicity are (a) *irreducibility* i.e. for each pair of states $i, j \in [N]$, there is an $s \in \mathbb{N}$ such that $p_{ij}^s > 0$ and (b) *aperiodicity* i.e., gcd$\{s : p_{ij}^{(s)} > 0\} = 1$ for all $i, j \in [N]$.

Time Reversibility

An ergodic Markov chain is said to be *(time-)reversible* iff either (and hence both) of the following equivalent conditions hold

1. For all $i, j \in [N]$, $p_{ij}\pi_i = p_{ji}\pi_j$ ("detailed balance" property)
2. The matrix $D^{1/2}PD^{-1/2}$ is symmetric, where $D^{1/2}$ is the diagonal matrix diag$(\pi_0^{1/2}, ..., \pi_{N-1}^{1/2})$ and $D^{-1/2}$ its inverse.

In the sequel only reversible states are going to be considered. Any ergodic reversible Markov chain can be represented by a weighted undirected graph G, the *underlying graph* of the chain.

Approaching Stationarity

- Consider the problem of sampling elements of the state space, assumed very large, according to the stationary distribution π.
- The desired distribution can be realized by picking an arbitrary initial state and simulating the transition of the Markov chain according to probabilities p_{ij}.
- As the number t of simulation steps increases, the distribution of the random variable X_t approaches π.
- The rate of convergence to stationarity is measured by the following time-dependent measure of deviation from the limit:

  Definition 1. *For any non-empty subset $U \subseteq [N]$, the* **relative pointwise distance** *(r.p.d.) over U after t steps is given by:*

$$\Delta_U(t) = max_{i,j \in U} \frac{|p_{ij}^{(t)} - \pi_j|}{\pi_j}$$

R.P.D. Discussion

The R.P.D. is quite strict for the following two reasons:

1. It demands that the distribution of the chain is close to the stationary distribution at *every* point
2. It demands that the convergence is rapid from *every* initial state

Rate of Convergence

Proposition 1. *Let P be the transition matrix of an ergodic reversible Markoc chain, π' its stationary distribution and $\{\lambda_i : 0 \leq i \leq N-1\}$ its (necessarily real) eigenvalues with $\lambda_0 = 1$. Then for any non-empty subset $U \subseteq [N]$ and all $t \in \mathbb{N}$, the relative pointwise distance $\Delta_U(t)$ satisfies*

$$\Delta_U(t) \leq \frac{\lambda_{max}^t}{\mathit{min}_{i \in U} \pi_i},$$

where $\lambda_{max} = \mathit{max}\{|\lambda_i| : 1 \leq i \leq N-1\}$.

Rate of Convergence

Proposition 2. *With the same notation and assumptions as in Proposition 1 the relative pointwise distance* $\Delta(t)$ *over* $[N]$ *satisfies*

$$\Delta(t) \geq \lambda_{max}^{t}$$

for all even $t \in \mathbb{N}$. *Moreover, if all eigenvalues of* P *are non-negative, the bound holds for all* $t \in \mathbb{N}$.

Rate of Convergence

Proposition 3. *With the same notation as in Proposition 1, suppose also that the eigenvalues of* P *are ordered so that* $1 = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{N-1} > -1$. *Then the modified chain with transition matrix* $P' = \frac{1}{2}(I_N + P)$, *where* I_N *is the* $N \times N$ *identity matrix, is also ergodic and reversible with the same stationary distribution, and its eigenvalues* $\{\lambda'_i\}$, *similarly ordered, satisfy* $\lambda'_{N-1} > 0$ *and* $\lambda'_{max} = \lambda'_1 = \frac{1}{2}(1 + \lambda_1)$

Conductance

Definitions

For any $S \subseteq [N]$ and $S \neq \varnothing$ with $\overline{S} = [N] \setminus S$ we define:

$$\Phi_S = \frac{F_S}{C_S}$$

where

- $C_S = \sum_{i \in S} \pi_i$ the *capacity* of S and
- $F_S = \sum_{i \in S, j \in \overline{S}} p_{ij} \pi_i$ the ergodic flow out of S.

Obviously $0 < F_S \leq C_S < 1$

INTUITION: Φ_S is the conditional probability that the stationary process crosses the cut from S to $\overline{S}$ in a single step given that it starts at S.

Conductance

Definition 2. *The* **conductance** *of a chain is defined to be:*

$$\Phi = \min_{0<|S|<N, C_S \leq 1/2} \Phi_S$$

For all sets S it holds that $F_S = F_{\overline{S}} = \sum_{i \in S j \in \overline{S}} w_{ij}$ while $C_S = 1 - C_{\overline{S}}$ and thus $\Phi_{\overline{S}} = \Phi_s \frac{C_S}{1-C_S}$ and therefore

$$\Phi = \min_{0<|S|<N} \max\{\Phi_s, \Phi_{\overline{S}}\}$$

Conductance and the Rate of Convergence

Lemma 1. *For an ergodic reversible Markov chain with underlying graph G, the second eigenvalue λ_1 of the transition matrix satisfies*

$$\lambda_1 \leq 1 - \frac{\Phi(G)^2}{2}$$

From Proposition 1 and Lemma 1 an upper bound on the distance from stationarity of a reversible chain, in terms of the conductance of the underlying graph can be set:

Theorem 1. *Let G be the underlying graph of an ergodic reversible Markov chain all of whose eigenvalues are non-negative, and π its stationary distribution. Then for any non-empty subset $U \subseteq [N]$ and all $t \in \mathbb{N}$, the relative pointwise distance satisfies*

$$\Delta_U(t) \leq \frac{\left(1 - \Phi(G)^2/2\right)^t}{\min_{i \in U} \pi_i}$$

Conductance and the Rate of Convergence

Lemma 2. *For an ergodic reversible Markov chain with underlying graph G, the second eigenvalue λ_1 of the transition matrix satisfies*

$$\lambda_1 \geq 1 - 2\Phi(G)$$

From Proposition 2 and Lemma 2 we yield the corresponding lower bound on the distance from stationarity of a reversible chain, in terms of the conductance of the underlying graph:

Theorem 2. *Let G be the underlying graph of an ergodic reversible Markov chain all of whose eigenvalues are non-negative, and suppose that $\Phi(G) \leq 1/2$. Then the relative pointwise distance $\Delta(t)$ over $[N]$ satisfies*

$$\Delta(t) \geq \big(1 - 2\Phi(G)\big)^t$$

for all $t \in \mathbb{N}$.

Rapid Mixing

Assume a *family* of ergodic Markov chains $MC(x)$ parametarised on strings $x \in \Omega \subseteq \Sigma^*$. For each $x \in \Omega$, let $\Delta^{(x)}(t)$ denote the r.p.d. of $MC(x)$ over its entire state space after t steps, and define the function $\tau^x : \mathbb{R}^+ \to \mathbb{N}$ by

$$\tau^{(x)}(\epsilon) = \min\Big\{t \in \mathbb{N} : \Delta^{(x)}(t') \leq \epsilon \text{ for all } t' \geq t\Big\}$$

Such family is called *rapidly mixing* iff there exists a polynomially bounded function $q : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{N}$ such that

$$\tau^{(x)}(\epsilon) \leq q(|x|, \lg \epsilon^{-1})$$

for all $x \in \Omega$ and $0 < \epsilon \leq 1$

A simulation of a rapidly mixing family should provide an efficient sampling scheme for the associated stationary distributions.

Rapid Mixing

Markov Chain Simulation Paradigm

if $R(x) = \varnothing$ **then**
    halt with output ?
else
    state=initial state of $MC(x)$
    for $t = 1$ to $q(|x|, \lg(2\epsilon)^{-1})$ **do**
        simulate one step of $MC(x)$
    end for
    if *state*$\in R(x)$ **then**
        halt with output *state*
    else
        halt with output ?
    end if
end if

Rapid Mixing

The above algorith, yields a f.p. almost uniform generator for R under the following conditions:

1. The construction problem of R is solvable in polynomial time (and therefore the non-emptiness of $MC(x)$ can be tested and an initial state found).
2. Individual steps of $MC(x)$ can be simulated in time polynomial in $|x|$ (therefore the MC has a simple local structure).
3. In the stationary distribution of $MC(x)$, the probability of being at an element $R(x)$ is bounded below by $1/p(|x|)$ for some polynomial p (so that the probability of failure is not that large)
4. q is polynomially bounded, i.e., the family is rapidly mixing. (Critical condition: the states are very close to stationarity after visiting only a very small proportion of their states)

Rapid Mixing

Assume that all states in some family are reversible and are known to have non-negative eigenvalues. Also assume that the minimum probability $\pi_{\min}^{(x)}$ assigned to any state in the stationary distribution of $MC(x)$ satisfies:

$$\lg \pi_{min}^{(x)^{-1}} \leq q'(|x|)$$

for all $x \in \Omega$ and some polynomial q'. Then:

Corollary 1. *Let $\{MC(x) : x \in \Omega\}$ be a family of ergodic reversible Markov chains, and let $G(x)$ be the underlying graph of $MC(x)$. Under the above assumptions, the family is rapidly mixing if and only if*

$$\Phi(G(x)) \geq 1/p(|x|)$$

for all $x \in \Omega$ and some polynomial p.

Rapid Mixing - Example

The following example confirms that for ergodic chains with negative eigenvalues, rapid mixing is not guaranteed by polynomial lower bound on the conductance.

Example 1. *Consider the family of chains $MC(n)$ parameterized on natural numbers $n \in \mathbb{N}^+$, in which $MC(n)$ has state space $V_1 \cup V_2$ for disjoint V_1, V_2 with $|V_1| = |V_2| = N/2$ and $N = 2^n$. Define the matrix $P = (p_{ij})$ by*

$$p_{kjk}$$

$$p_{ij} = \begin{cases} 2/N, & \textit{if } i \in V_1, j \in V_2 \textit{ or } i \in V_2, j \in V_1 \\ 0, & \textit{otherwise} \end{cases}$$

and let the transition matrix of $MC(n)$ be $(1-\alpha)P + \alpha I_N$. The chain $MC(n)$ is obviously ergodic for $0 < \alpha < 1$ since it is symmetric and it is also reversible with uniform stationary distribution.

Consider the non-empty subset $S = A \cup B$ of states with $A \subseteq V_1$ and $B \subseteq V_2$ and $|S| \leq N/2$. Let $a = |A|/N$ and $b = |B|/N$ then:

1. $C_S = a + b$

2.

$$\begin{aligned} F_S &= \sum_{i\in S, j\in \overline{S}} \pi_i p_{ij} = \frac{1}{N} \sum_{i\in S, j\in \overline{S}} p_{ij} \\ &= \frac{1}{N}\frac{2(1-\alpha)}{N}\big(|A|(N/2-|B|)+|B|(N/2-|A|)\big) \\ &= \frac{2(1-\alpha)}{N}\big(a(N/2-bN)+b(N/2-aN)\big) \end{aligned}$$

From (1) and (2) we obtain that

$$\Phi(s) \equiv \frac{F_S}{C_S} = (1-\alpha)\big(1-\frac{4ab}{a+b}\big)$$

Under the constraint $a+b \leq 1/2$, Φ_S *is minimized for* $a=b=1/4$ *and thus*

$$\Phi(G(n)) = \min_S \Phi_S = (1-\alpha)/2$$

Rapid Mixing - Example

For $t = 2m + 1$ and $m \in \mathbb{N}$ $pr = Pr(X_t \in V_1 | X_0 \in V_1) = 1 - (1 - \alpha)^t$

Assume that $\alpha = N^{-1}$ and $t = f(n)$ where $f : \mathbb{N} \to \mathbb{R}$ a polynomial. Then $pr \to 0$ as $n \to \infty$, meaning that the family is not rapidly mixing despite the fact that the conductance is large.

Note that the above chains fail to converge fast because they are "almost periodic"

Discussion

How about variational distance??

The *variational distance* from initial state i is defined by

$$\Delta_i^{var}(t) = \frac{1}{2}\sum_j |p_{ij}^{(t)} - \pi_j|$$

1. Φ also characterizes the rapid mixing property when formulated in terms of variation distance.
2. In the v.d. formulation the initial state plays a significant role. In particular the upper bound depends only on the stationary probability of the initial state, rather than on π_{min}.
3. The lower bound does not rule out the possibility that a chain may converge fast in variation distance from certain initial states even when the conductance is small.

Discussion

Many natural Markov chains can be viewed as a simple random walk on a graph $H(V, E)$. Transitions in H are made as follows:

1. any vertex u goes to an adjacent vertex with probability β/d (where d is the maximum degree of H and $\beta \leq 1$ a positive constant).
2. Any vertex u has a self-loop probability $1 - \beta\text{deg}(u)/d$

If H is connected and $\beta < 1$ then the chain is ergodic; Additionally it is reversible with uniform stationary distribution (double stochastic matrix).

For any subset $S \subseteq V$, let $\Gamma(S)$ denote the *cut set* in H defined by S, and define the *(edge) magnification* $\mu(H)$ of H to be:

$$\mu(H) = \min_{0<|S|\leq|V|/2} \frac{|\Gamma(S)|}{|S|}$$

Obviously $0 < \mu(H) \leq d$

Discussion

Proposition 4. *Let G be the underlying graph of an ergodic random walk on a graph H with maximum degree d and transition probabilities β/d between distinct adjacent states. Then the conductance of G is given by*

$$\Phi(G) = \beta\mu(H)/d$$

Example

- Consider relation $B : \mathbb{N} \rightarrow \{0,1\}^n, n \in \mathbb{N}$.
- Construct a family of Markov chains $MC(n)$ to be used as an almost uniform generator for B.
- Consider the random process that moves through the state space of $B(n)$ by flipping a random bit in each transition.
- The above allows to view $MC(n)$ as a random walk an the $n-$regular graph $H(n)$ with vertex set $B(n)$ and edge set

$$\{(u,v) \in B(n) \times B(n) : D(u,v) = 1\}$$

  Obviously $H(n)$ defines the n-dimensional hypercube.
- By introducing self-transitions with probability $1/2$ we obtain a Markov chain with uniform stationary distribution.

Example

Theorem 3. *The magnification of the n-dimensional hypercube satisfies $\mu(H(n)) \geq 1$.*

Corollary 2. *The above family of Markov chains is rapidly mixing.*

Proof:

1. $\Phi(G) = \beta\mu(H)/d = \frac{1}{2}\frac{\mu(H)}{n} \geq \frac{1}{2}\frac{1}{n} = \frac{1}{2n}$
2. The minimum stationary probability satisfies: $\pi_{min} = \frac{1}{2^n} \Rightarrow \lg \pi_{min}^{(n)} = -n$

So the conditions of the corollary 1 are satisfied and therefore the Markov chain is rapidly mixing.

Example

Proof of Theorem 3

- Let $N = 2^n$ the set of states of $MC(n)$
- Specify a canonical path in $H(n)$ between every ordered pair in $H(n)$
- No oriented edge on $H(n)$ is contained to more than bN of the paths.
- Let $S \subseteq N$ and $0 < |S| \leq N/2$
- The number of paths crossing the cut from S to $\overline{S}$ is: $|S|(N - |S|) \geq |S|N/2$
- So the number of cut edges is: $|\Gamma(S)| \geq \frac{|S|N}{2bN} = \frac{|S|}{2b}$
- $\mu(H(n)) = \min_S \frac{|\Gamma(S)|}{|S|} \geq \frac{|S|}{2b|S|} = \frac{1}{2b}$
- Goal: Define the set of canonical paths that are sufficiently edge disjoint so that the $\mu(H(n))$ can be bounded from below.

Example

Proof of Theorem 3 (Cont'd)

- Let $u = (u_i)_{i=0}^{N-1}$ and $v = (v_i)_{i=0}^{N-1}$ distinct elements of $B(n)$.
- Let $i_1 < ... < i_l$ the positions in which u, v differ.
- Then for $1 \leq j \leq l$ the j-edge for a canonical path from u to v corresponds to a transition with the j-th bit flipped from u_{i_j} to v_{i_j}.
- Consider an arbitrary transition t of $MC(n)$ and denote with $P(t)$ the set of paths that contain t, viewed as ordered pairs of states.
- Suppose t takes state $w = (w_i)$ converting it to $w' = (w_i')$by flipping the value of w_k.
- Define injective mapping $\sigma_t : P(t) \rightarrow B(n)$ as follows: $\sigma_t(u, v) = (s_i)$ with

$$s_i = \begin{cases} u_i, & 0 \leq i \leq k \\ v_i, & k < i < n \end{cases}$$

Example

Proof of Theorem 3 (Cont'd)

- σ_t is indeed *injective*. Consider:

$$u_i = \begin{cases} s_i, & 0 \le i \le k \\ w_i, & k < i < n \end{cases} \qquad v_i = \begin{cases} w'_i, & 0 \le i \le k \\ s_i, & k < i < n \end{cases}$$

- $\sigma_t : P(t) \to B(n)$ is injective $\Rightarrow |P(t)| \le N$
- All vectors (s_i) satisfy that $s_k = w_k \Rightarrow |P(t)| \le |B(n)|/2 = N/2$
- Set the value of parameter $b = 1/2 \Rightarrow \mu(H(n)) \ge 1$.

Summary

- Definition of the Conductance Φ of a Markov chain.
- Definition of the notion of a rapidly mixing Markov chain.
- How is conductance related to the rapid mixing property of the Markov chain.
- Indicative example using canonical paths.

Problem $P1$: Estimating the Number of Perfect Matching in a Bipartite Graph G

- The decision problem (determining whether a graph has at least one perfect matching) is in $\mathbf{P}$.
- The problem of counting the number of perfect matchings in a given bipartite graph is $\#\mathbf{P}$- complete.
- The problem $P1$ is equivalent to the problem of computing the permanent of a 0-1 matrix ($O(n2^n)$).
- $P1$ can be reduced to sampling uniformly at random from all the perfect matchings in G.
- The problem of random generations is NOT substantially easier than the problem of counting.
- It suffices to generate a perfect matching "almost uniformly" from all perfect matchings in G.

- “Almost Uniform Generation” can be done by simulating a random walk on a Markov chain derived from the input graph G (which is NOT the same as the simple random walk on G).

Problem $P1$: Reduction to Uniform Generation

- Let M_k be the set of distinct matches of size k in G. Then: $m_k = |M_k|$. The goal is to estimate $m_n = |M_n|$.
- A *uniform generator* $\mathcal{U}_k$ for M_k is a randomized polynomial time algorithm that takes G as input and outputs a matching $m \in M_k$ such that m is uniformly distributed over M_k.
- **Claim:** $\mathcal{U}_k$ can be used to get an (ϵ, δ)-FPRAS from m_k.
- The idea is to use *self-reducibility* a randomized reduction of a problem of size i to the same problem of size $i - 1$.
- Use $\overline{\mathcal{U}_k}$ to obtain a near uniform generator form $M_k \cup M_{k-1}$.
- Estimate the ratio $r_k = \frac{m_k}{m_{k-1}}$ with $1 \leq k \leq n$ and $r_1 = m_1$.
- Then $m_n = \prod_{i=1}^{n} r_i$

Problem $P1$: Reduction to Uniform Generation

- Let $\alpha \geq 1, \alpha \in \mathbb{R}$ such that $1/\alpha \leq r_k \leq \alpha$. Then taking $N = n^7\alpha$ samples uniformly at random from $M_k \cup M_{k-1}$ guarantees that with probability at least $1 - c^n$ (for $c > 1$) : $(1 - 1/n^3)r_k \leq \widehat{r_k} \leq (1 + 1/n^3)r_k$
- If $\alpha \leq n^2$ and $\overline{U_k}$ for $M_k \cup M_{k-1}$ with error $\rho \leq 1/n^4$ then there is an (ϵ, δ)-FPRAS for M_n.

  Theorem 4. *Let G be a bipartite graph with minimum degree at least $n/2$. Then, for all k, $1/n^2 \leq r_k \leq n^2$.*

- Therefore for bipartite graphs with minimum degree $n/2$ $\alpha \leq n^2$.

Problem $P1$: Near-Uniform Generation of Matchings

Define the underlying graph $H(n)$

Fix a bipartite graph $G(V, E)$ with minimum degree $n/2$.

GOAL: Find a near-uniform generator for $M_k \cup M_{k-1}$ that has error $\rho \leq 1/n^4$.

Consider a Markov chain $MC(k)$ with states in $M_k \cup M_{k-1}$.

Execution of $MC(k)$ is equivalent to execution of a random walk on $H(k)$. For example for $k = n$ the random walk on the graph $H(n)$ is used to estimate $|M_n|/|M_{n-1}|$.

The vertices of H_n are $V_H = M_n \cup M_{n-1}$

The set of transitions (along with transition probabilities) in $H(n)$ are defined as follows:

In any state $m \in MC(n)$ with probability $1/2$ remain in the same state. Otherwise with probability $1/2$ choose an edge $e = (u, v) \in E$ uniformly at random and select one of the following cases:

- **Reduce:** If $m \in M_n$ and $e \in m$ move to the state (matching) $m' = m - e$
- **Augment:** if $m \in M_{n-1}$, with u, v unmatched in m, move to $m' = m + e$
- **Rotate:** if $m \in M_{n-1}$, with u matched to w and v unmatched in m, move to $m' = (m + e) - f$ where $f = (u, w)$ (symmetrical rotation for the case where v matched and u unmatched)
- **Idle:** otherwise stay at the current state

Theorem 5. *The Markov chain $MC(n)$ is ergodic and its stationary distribution is uniform on $M_n \cup M_{n-1}$.*

It remains to be shown that $MC(n)$ is rapid mixing. Since error $\rho \leq 1/n^4$ we also need $\Delta(t) \leq 1/n^4$

But $\Delta(t) \leq \frac{\lambda_{max}^t}{\min_{i\in[N]} \pi_i} = N\lambda_{max}^t$

Additionally $e^{-(1-\lambda_{max})} \geq 1 + (-1 + \lambda_{max}) = \lambda_{max}$ because for every $t \in \mathbb{R} e^t \geq 1 + t$

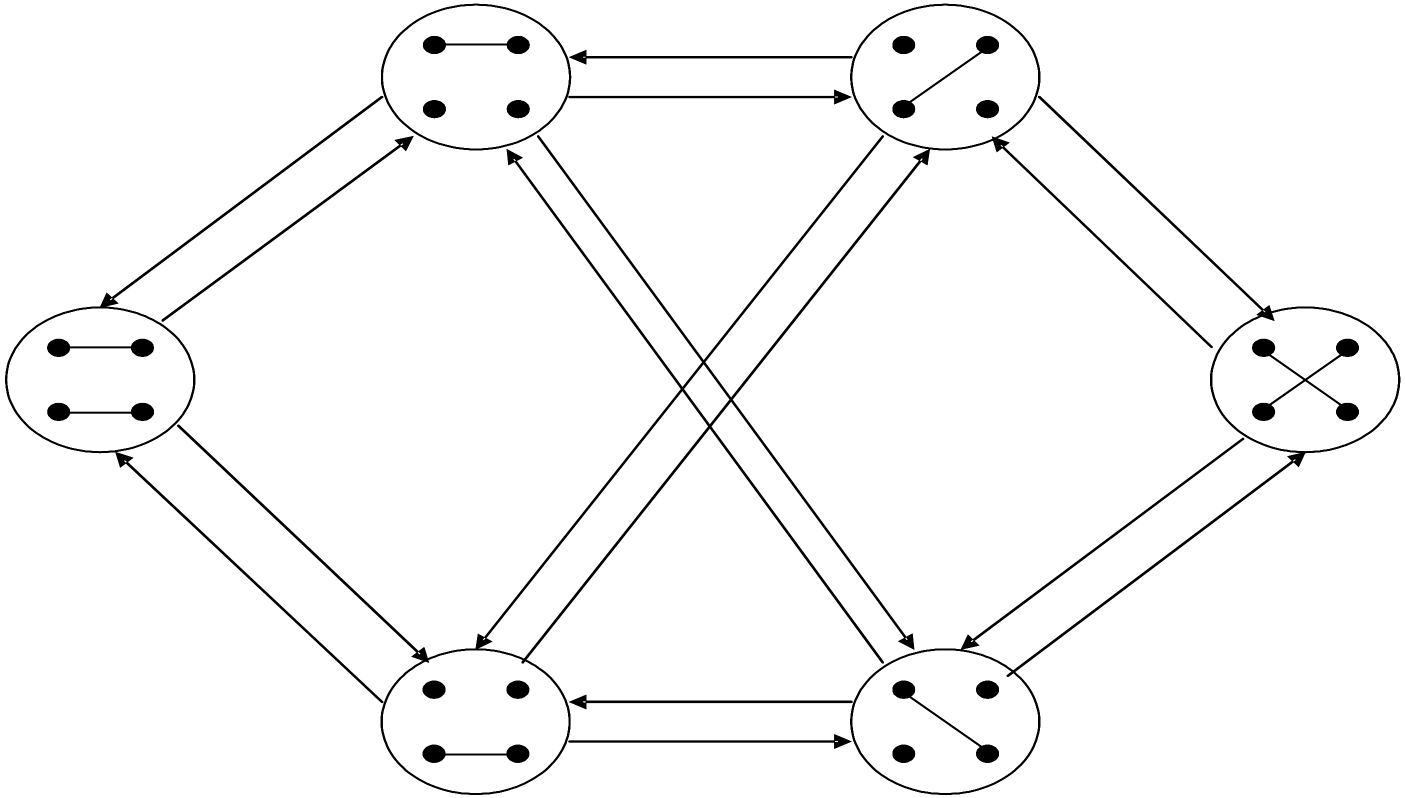
For t being $ln(n^4 N)/(1 - \lambda_{max}) \Rightarrow \Delta(t) \leq 1/n^4$ but for rapid mixing we need to show that $1/(1 - \lambda_{max})$ be bounded by a polynomial in n.

Use of Conductance

$$\lambda_{max} \leq 1 - \frac{\Phi^2}{2} \Leftrightarrow 1 - \lambda_{max} \geq \frac{\Phi^2}{2}$$
$$\Leftrightarrow \frac{1}{1 - \lambda_{max}} \leq \frac{2}{\Phi^2}$$

Thus it suffices to prove that $1/\Phi$ is bounded, then $1/(1 - \lambda_{max})$ is bounded as well.

Theorem 6. *For the Markoc chain* $MC(n)$ $\Phi \geq 1/12n^6$



Example of $MC(2)$ for $G = K_{2,2}$

Problem $P1$: Near-Uniform Generation of Matchings

Bound the conductance Φ

- Let N the set of states of $MC(n)$
- Specify a canonical path in $H(n)$ between every ordered pair in $H(n)$
- No oriented edge on $H(n)$ is contained to more than bN of the paths.
- Let $S \subseteq N$ and $0 < |S| \leq N/2$
- The number of paths crossing the cut from S to $\overline{S}$ is: $|S|(N - |S|) \geq |S|N/2$
- So the number of cut edges is: $|\Gamma(S)| \geq \frac{|S|N}{2bN} = \frac{|S|}{2b}$
- $\mu(H(n)) = \min_S \frac{|\Gamma(S)|}{|S|} \geq \frac{|S|}{2b|S|} = \frac{1}{2b}$
- Goal: Define the set of canonical paths that are sufficiently edge disjoint so that the $\mu(H(n))$ can be bounded from below.

Problem $P1$: Near-Uniform Generation of Matchings

Cannonical Paths

- *Path Type A:* For every node $s \in M_{n-1} \cup M_n$ associate a *partner* node $\overline{s} \in M_n$ and choose a canonical path between s and $\overline{s}$. (Obviously if $s = \overline{s}$ the canonical path is empty)
- *Path Type B:* Specify also canonical paths for all pairs of nodes in M_n
- A canonical path between $s, y \in M_{n-1} \cup M_n$ consists of three segments $s \to \overline{s} \to \overline{t} \to t$

Problem $P1$: Near-Uniform Generation of Matchings

Specifying Type A Paths

- **Claim:** In G every near-perfect matching has an augmenting path of length at most 3.
- *Case 1:* $s \in M_n \Rightarrow \overline{s} \in M_n \Rightarrow s = \overline{s}$ (empty path)
- *Case 2:* $s \in M_{n-1}$ and there exists $e : s + \{e\} \in M_n$ (Path of length 1)
- *Case 3:* $s \in M_{n-1}$ but no direct augmentation. (Due to claim) $s \to \overline{s}$ due a **rotate** followed by **augment**
- Let $m \in M_n$ and $\kappa(m) = \{s \in M_n \cup M_{n-1} | \overline{s} = m \text{ and } s \neq m\}$
- For any $m \in M_n$, $|\kappa(m) \leq n^2|$
- Proof (sketch): Consider $s \in M_{n-1}$, with $\overline{s} = m$. There are two cases $s \to m$ through a path of length 1 or 2. There are $n + n(n-1)$ such different near-perfect matchings.

Problem $P1$: Near-Uniform Generation of Matchings

Specifying Type B Paths

- Fix $s, t \in M_n$ and let $d = s \oplus t$ the symmetric difference of the edges of the two perfect cycles.
- d can be decomposed into a collection of *disjoint*, *even-length* and *alternating cycles*
- The length of each cycle is at least 4
- Assume the set of even cycles totally ordered and a specified vertex designated as the *start-vertex*
- Transform s to t by performing local changes *unwindings* of the cycles in d in the specified order of the cycles.
- These local changes can be seen as transitions on $H(n)$. Thus a path from $s \rightarrow t$ is formed this way.

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Theorem 7. *Any transition T in H lies on at most $3n^4N$ distinct canonical paths.*

Proof: Fix any transition $T = (u, v)$ in the graph H.

1. T lies on segment type A: ($s \to \overline{s}$ or $t \to \overline{t}$).
   If T lies in path $s \to \overline{s}$ then $v \subseteq \overline{s}$. Additionally there exists at most one perfect matching of which v is subset. This perfect matching is $\overline{v} = \overline{s}$ and thus $s \in \kappa(\overline{v})$.
   Similar arguments for having $t \in \kappa(\overline{u})$.
   The total number of pairs s, t for which T lies on a type A segment of the canonical path is bounded by:

$$(|\kappa(\overline{v}| + |\kappa(\overline{u}|)N \leq 2n^2N$$

2. T lies on the segment of type B: ($\overline{s} \to \overline{t}$)

Let $CP(T)$ be the pairs of perfect matchings whose canonical path contains T.

The number of canonical paths whose type B segments contain T are:
$|\kappa(\overline{s}||CP(T)|)|\kappa(\overline{t}| \leq n^4|CP(T)| \leq n^4 N$ since $|CP(T)| \leq N$