Proof Theoretical Framework and Term Equations

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pp. 82 – 103 from the Apt's book.

From the previous lectures

- Equivalence of CSPs
- Constraint solvers
  - Complete constraint solver
  - Incomplete constraint solver

A proof theoretical framework

- Used to define complete constraint solvers
- Proof rules (equivalence preserving)
  - Domain reduction rules
  - Transformation rules

Proof rules

Let us assume that ϕ and ψ are CSPs. A proof rule

$$\frac{\phi}{\psi} \text{ or } \frac{\langle \mathcal{C};\mathcal{DE}\rangle}{\langle \mathcal{C}';\mathcal{DE}'\rangle}$$

- is equivalence preserving if ϕ and ψ are equivalent.
- is consistency preserving if ϕ and ψ are both either consistent or inconsistent

Domain reduction rules

Assume that $\phi := \langle \mathcal{C}; \mathcal{DE} \rangle$ and $\psi := \langle \mathcal{C}'; \mathcal{DE}' \rangle$

- $\mathcal{DE} := x_1 \in D_1, \ldots, x_n \in D_n$
- $\mathcal{DE}' := x_1 \in D'_1, \ldots, x_n \in D'_n$
- $\forall i \in [1, n] : D'_i \subseteq D_i$
- $\mathcal{C}'$ is obtained from $\mathcal{C}$ by restricting each constraint to the correspondin subsequence of the domains $D'_1, \ldots, D'_n$
- failure as empty set

Domain reduction rules

Linear Disequality:

Our old friend who deals with linear inequalities over integer intervals:

$$\frac{\langle x < y; x \in [l_x, h_x], y \in [l_y, h_y] \rangle}{\langle x < y; x \in [l_x, min(h_x, h_y - 1)], y \in [max(l_y, l_x + 1), h_y] \rangle}$$

Domain reduction rules

Equality:

$$\frac{\langle x{=}y ; x{\in}D_x , y{\in}D_y \rangle}{\langle x{=}y ; x{\in}D_x{\cap}D_y , y{\in}D_x{\cap}D_y \rangle}$$

- The constraint is solved if
  - $D_x \cap D_y = \emptyset$
  - $D_x \cap D_y$ is a singleton set

Domain reduction rules

Disequality:

$$\frac{\langle x \neq y; x \in D_x, y = a \rangle}{\langle ; x \in D - a, y = a \rangle}$$

- Failure if $D_x - \{a\} = \emptyset$

Transformation rules

- $\frac{\langle \mathcal{C}; \mathcal{DE} \rangle}{\langle \mathcal{C}'; \mathcal{DE}' \rangle}$
- $\mathcal{C}' \neq \emptyset$
- $\mathcal{DE}'$ extends $\mathcal{DE}$
- failure when $\perp$ generated

Transformation rules

Disequality transformation:

$$\frac{\langle s \neq t; \mathcal{DE} \rangle}{\langle x \neq t, x = s'; \mathcal{DE}', x \in \mathcal{Z} \rangle}$$

- s is not a variable
- $\mathcal{DE}$ includes all the variables of s and t
- x doesn't appear in $\mathcal{DE}$

Transformation rules

Variable elimination:

$$\frac{\langle \mathcal{C}; \mathcal{D}\mathcal{E}, x = a \rangle}{\langle \mathcal{C}\{x/\bar{a}\}; \mathcal{D}\mathcal{E}', x = a \rangle}$$

- $\{x/\bar{a}\}$ is a substition
- $\bar{a}$ stands for the constant that denotes the value of a in our language of constraints

Example:

$$\frac{\langle 3xy^2 + 5xy - 5yz \leq 6; x \in [0,100], y = 2, z \in [0,100] \rangle}{\langle 3 \cdot x \cdot 4 + 5 \cdot x \cdot 2 - 5 \cdot 2 \cdot z \leq 6, x \in [0,100], y = 2, z \in [0,100] \rangle}$$

Transformation rules

Resolution:

$$\frac{\langle C_1 \vee L, C_2 \vee \bar{L} \rangle}{\langle C_1 \vee L, C_2 \vee \bar{L}, C_1 \vee C_2 \rangle}$$

Introduction rules in general:

$$\frac{\langle \mathcal{C}; \mathcal{DE}, \rangle}{\langle \mathcal{C}, C; \mathcal{DE}, x = a \rangle}$$

- new constraint C is introduced

Applying rules

Let $\mathcal{P} = \langle \mathcal{C} \cup \mathcal{C}_1; \mathcal{DE}, \mathcal{DE}_1 \rangle$ be a CSP and $\frac{\langle \mathcal{C}_1; \mathcal{DE}_1, \rangle}{\langle \mathcal{C}_2; \mathcal{DE}_2 \rangle}$ rule (R) that we are trying to apply on $\mathcal{P}$.

- Variable that appears in the conclusion but not in the premise is called $i$ntroduced variable Application of the R to the CSP $\mathcal{P}$:
- R is applied on $\mathcal{P}$
  1. **renaming** introduced variables
  2. **replacing** $\langle \mathcal{C}_1; \mathcal{DE}_1, \rangle$ with $\langle \mathcal{C}_2; \mathcal{DE}_2 \rangle$
  3. **restricting** the constraint of $\mathcal{C}$ to the domains of $\mathcal{DE}, \mathcal{DE}_2$

Applying rules

- When **equivalence preserving rule** (R) is applied on ϕ to get ψ, ϕ and ψ are equivalent
- if ϕ differs from ψ, the application of R was **relevant**
- CSP ϕ is **closed under the application of** R if one of the following holds:
  - R cannot be applied to ϕ
  - application of R is not relevant

Derivations

Derivation is a sequence of CSPs obtained by applying the proof rules. A finite derivation is called:

- **successful** if the last CSP is the first solved CSP
- **failed** if last CSP is the first failed CSP
- **stabilising** if the last CSP is closed under the rules

Term equation

Alphabet consists of

- fixed and infinite set of **variables**
- set of **functions**, each function has a fixed arity (possibly zero)
- "(", ")" and ","

Terms are defined recursively

- a variable is a term
- a function that is applied on terms is a term

Substitition

Finite mapping from variables to terms

$$\{x_1/t_1, \ldots, x_n/t_n\}$$

- variables: $x_1, \ldots x_n$ – $Dom(\theta) = \{x_1, \ldots, x_n\}$
- terms: $t_1, \ldots t_n$ – $Range(\theta) = \{t_1, \ldots, t_n\}$
- binding: x_i/t_i
- $\forall i \in [1, n] : x_i \neq t_i$

Substitutions

Applying substitution θ to the term s

- $s\theta$ (instance of s)
- simultaneously for each binding of θ:
  - in the term s, replace variable x_i with term t_i

Example: language of arithmetic expressions

$$((1 + x) + y - 1)\{x/y\} \equiv ((1 + y) + y - 1)$$

Composite substitutions

Let θ and η be substitutions. **Composition** of θ and η:

$$(\theta\eta)(x) := (x\theta)\eta$$

θ is **more general than** τ if for some substitution η

$$\tau = \theta\eta$$

e.g. $\theta := \{y/g(x, a), z/b\}, \tau := \{x/c, y/g(c/a), z/b\}$

Unification

- θ **unifies** s and t iff $s\theta \equiv t\theta$
- θ is a **mgu** of s and t iff
  1. $s\theta \equiv t\theta$
  2. θ is more general than all unifiers of s and t

Example: {y/g(x,a),z/b} is an mgu of $f(g(x,a),z)$ and $f(y,b)$

Unification

- θ is a unifier of $E := \{s_1 = t_1, \ldots, s_n = t_n\}$ if

$$s_1\theta \equiv t_1\theta, \ldots, s_n\theta \equiv t_n\theta$$

Solved form:

E is in solved form iff $\forall i \in [1, n] : x_i \notin Var(t_i)$ and x_i is not elsewhere in E than on the left hand side of the $x_i = t_i$

Strong mgu

Call an mgu θ of of a set of equations E **strong** if for every unifier η of E we have $\eta = \theta\eta$.

Lemma: If $E := \{s_1 = t_1, \ldots, s_n = t_n\}$ is in **solved form** then $\theta := \{s_1/t_1, \ldots, s_n/t_n\}$ is a **strong mgu**.

Unif. prob. as CSP

- τ set of all terms
- $s = t \mapsto \{(x_1\eta, \ldots .x_n\eta) | \eta$ unifies s and $t\}$

Example: $x = f(y)$

UNIF proof system...

Decomposition

$$\frac{f(s_1, \ldots s_n) = f(t_1, \ldots t_n)}{s_1 = t_1, \ldots s_n = t_n}$$

Failure 1

$$\frac{f(s_1, \ldots s_n) = g(t_1, \ldots t_n)}{\bot}, \ f \neq g$$

Deletion

$$\frac{x = x}{}$$

...*UNIF* proof system

Transposition

$$\frac{t=x}{x=t},\ t \text{ is not a variable}$$

Substitution

$$\frac{x=t, E}{x=t, E\{x/t\}},\ x \notin Var(t) \wedge x \in Var(E)$$

Failure 2

$$\frac{x=t}{\perp},\ \text{where } x \in Var(t) \text{ and } x \not\equiv t$$

Example

UNIF

Theorem *Consider a failed or a stabilising derivation in the UNIF system, starting eith a finite set of equtions E and terminating with a set of constraints F. If E has a unifier then F is a solved form that determines an mgu of E . If there is no unifier for E then F contains* $\perp$.

Summary

- domain reduction– *vs.* transformation rules
- term equation
- substitution and mgu
- unification probles and CSP
- *UNIF* system
- applications?

kotitehtävä 1

Esitä jokin kokonaislukudomaineihin ja rajoitteisiin perustuva CSP, jolla on

a) Stabiloituva, epäonnistuva johto, jonka pituus on vähintään 4 askelta

b) Onnistunut johto, jonka pituus on vähintään 4 askelta

c) Äärettömän pituinen johto

Esitä jokaisesta kohdasta käyttämäsi sääntöjoukko (Esimerkiksi esityksen alussa olleet muokkaussäännöt domainia pienentävät säännöt ovat sopiva joukko kaikkiin kohtiin), CSP ja pyydetty johto sille (c– kohdassa riitää, että esität, miten johto konstruoidaan). Jokaisessa johdossa on käytettävä vähintään kahta eri sääntöä.

Johdossa saa käyttää peruskoulusta tuttua aritmetiikkaa Älä palauta kirjan esimerkkejä.

kotitehtävä 2

a) Olkoon $E := \{f(g(x, a), z) = f(y, b)\}$. Etsi E:n mgu, jos sellainen on olemassa käyttäen *UNIF* sääntöjoukkoa. Jos unifioijaa ei ole olemassa, niin todista se.

b) Olkoon

- $t_1 = f_2(f_1(g(x, a)), x)$
- $t_2 = f_2(f_1(y), b)$

Etsi termien t_1 ja t_2 mgu, jos sellainen on olemassa käyttäen *UNIF* sääntöjoukkoa. Jos unifioijaa ei ole olemassa, niin todista se.