Seminar Presentation on Constraint Programming
T-79.194 Tietojenkäsittelyteorian seminaari

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2. Linear inequalities over reals

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$$
\begin{array}{ccccccccccc}
-x_1 & + & x_2 & + & x_3 & + & 2x_4 & - & x_5 & = & 4 \\
x_1 & + & x_2 & - & x_3 & - & 4x_4 & + & 3x_5 & = & 0 \\
x_1 & - & x_2 & + & x_3 & & & - & 3x_5 & = & 2
\end{array}
\begin{array}{r} (1) \\ (2) \\ (3) \end{array}
$$

$$x_1 = x_2 + x_3 + 2x_4 - x_5 - 4$$

Let's use the *SUBSTITUTION* rule with the other two equations as E.

$$\begin{array}{rcrcrcrcrcrcr} x_1 & = & x_2 & + & x_3 & + & 2x_4 & - & x_5 & - & 4 & & \\ & & 2x_2 & & & - & 2x_4 & + & 2x_5 & = & 4 & & \\ & & & & 2x_3 & + & 2x_4 & - & 4x_5 & = & 6 & & \end{array} \begin{array}{r} (4) \\ (5) \\ (6) \end{array}$$

$$x_2 = x_4 - x_5 + 2$$

Now we use the *SUBSTITUTION* rule with the third equation only as E.

$$\begin{array}{rcrcrcrcrcrcl} x_1 & = & x_2 & + & x_3 & + & 2x_4 & - & x_5 & - & 4 \\ & & x_2 & = & & & x_4 & - & x_5 & + & 2 \\ & & & & 2x_3 & + & 2x_4 & - & 4x_5 & = 6 & \end{array} \qquad \begin{array}{r} (7) \\ (8) \\ (9) \end{array}$$

$$x_3 = -x_4 + 2x_5 + 3$$

Let's apply *SUBSTITUTION* rule with the empty set of equations as E.

$$
\begin{aligned}
x_1 &= x_2 + x_3 + 2x_4 - x_5 - 4 && (10)\\
x_2 &= x_4 - x_5 + 2 && (11)\\
x_3 &= -x_4 + 2x_5 + 3 && (12)
\end{aligned}
$$

Backward substitution phase:

Using the *SUBSTITUTION* rule with the last equation as the selected one and the other two equations as E we transform the above set into

$$
\begin{aligned}
x_1 &= x_2 + x_4 + x_5 - 1 && (13)\\
x_2 &= x_4 - x_5 + 2 && (14)\\
x_3 &= -x_4 + 2x_5 + 3 && (15)
\end{aligned}
$$

Using now the *SUBSTITUTION* rule with the second equation as the selected one and the first one as E we transform the above set into

$$
\begin{aligned}
x_1 &= 2x_4 + 1 && (16)\\
x_2 &= x_4 - x_5 + 2 && (17)\\
x_3 &= -x_4 + 2x_5 + 3 && (18)
\end{aligned}
$$

At this moment no application of a rule of *LIN* is relevant and the derivation terminates with the last set of equations.

Theorem 4.37 (Gaussian Elimination) *The* Gaussian Elimination *algorithm always terminates. If the original finite sequence of linear equations E has a solution, then the algorithm terminates with a set of linear equations in a solved form that determines an mgu of E and otherwise it terminates with a set containing the false constraint* $\perp$.

3 Linear inequalities over reals

3.1 Syntax

By a *linear inequality over reals* we mean a constraint of the form

$$s \leq t,$$

where s and t are linear expressions. For example,

$$4x - 3.5y - 1.2z \leq 3x - 1.2 \cdot (2 + 2.5y + z) - 2x + 5$$

is a linear inequality.

Definition 4.38 Assume a predefined ordering $\prec$ on the variables. Fix a variable x.
We say that a linear inequality is in an x-**normal form** if it is in one of the following forms, where $x \ni Var(t)$ and t is a linear expression in normal form:

- the ${}^{\leq}x$ **-normal form:** $t \leq x$,
- the $x^{\leq}$ **-normal form:** $x \leq t$,
- the $\bar{x}$ **-normal form:** $t \leq 0$.

Example 4.39 Reconsider the linear inequality

$$4x - 3.5y - 1.2z \leq 3x - 1.2 \cdot (2 + 2.5y + z) - 2x + 5$$

It normalises to the $x^{\leq}$-normal form $x \leq \frac{0.5}{3}y + \frac{2.6}{3}$ and to the ${}^{\leq}y$-normal form $6x - 5.2 \leq y$. In contrast, its z-normal form is $6x - y - 5.2 \leq 0$, which is a $\bar{z}$-normal form, without an occurrence of z.

3.2 Linear inequalities and CSPs

Let's denote normal form by *NF*.

We say that two linear inequalities li_1 and li_2 with the same sequence of variables $x_1, ..., x_n$ are **equivalent** if the CSPs $\langle li_1; x_1 \in NF, ..., x_n \in NF \rangle$ and $\langle li_2; x_1 \in NF, ..., x_n \in NF \rangle$ are equivalent.
Further, we say that two linear inequalities li_1 and li_2 with a sequence X of common variables are **equivalent w.r.t.** X if the corresponding CSPs determined by these two inequalities are equivalent w.r.t. X.
In particular, the already discussed inequality

$$4x - 3.5y - 1.2z \leq 3x - 1.2 \cdot (2 + 2.5y + z) - 2x + 5$$

and each of its normal forms, so $x \leq \frac{0.5}{3}y + \frac{2.6}{3}$,$6x - 5.2 \leq y$, and $6x - y - 5.2 \leq 0$ are equivalent w.r.t. x, y.

3.3 The *INEQ* proof system

To reason about finite sets of linear inequalities we introduce now three proof rules. To present them we need the following notation.

Definition 4.40 Given a variable x and a set of linear equations LI we introduce the following three sets of linear inequalities derived from it:

$\bullet {}^{\leq}x(LI) := \{\ li \mid li$ is the ${}^{\leq}x$-normal form of an inequality from $LI\}$,

$\bullet x^{\leq}(LI) := \{\ li \mid li$ is the $x^{\leq}$-normal form of an inequality from $LI,\}$

$\bullet \bar{x}(LI) := \{\ li \mid li$ is the $\bar{x}$-normal form of an inequality from $LI\}$.

We also introduce the following operation on sets of linear inequalities:

$$E \cdot F := \{s \leq v | s \leq t \in E, t \leq v \in F\}.$$

So if either E or F is empty, then so is $E \cdot F$.

The idea behind the algorithm is simple. We repeatedly select a variable , say x, and perform the following steps:

• we normalise all inequalities in the current set LI to the x-normal form,

• we eliminate all occurrences of x by replacing the sets ${}^{\leq}x(LI)$ and $x^{\leq}\ (LI)$ by their 'composition' ${}^{\leq}x(LI)\ \cdot x^{\leq}\ (LI)$.

This procedure can be described using the following proof rule:

$$\frac{\overset{x - ELIMINATION}{LI}}{{}^{\leq}x(LI) \cdot x^{\leq}(LI), \bar{x}(LI)}$$

Additionally, we introduce the following two rules that deal with specific $\bar{x}$-normal forms:

$$\frac{\overset{DELETION}{s \leq t}}{}$$

if $s \leq t$ normalises to $r \leq 0$, where r is a strictly negative real or 0,

$$\frac{\overset{FAILURE}{s \leq t}}{\perp}$$

if $s \leq t$ normalises to $r \leq 0$, where r is a strictly positive real.

Denote the set of these three rules by *INEQ*.

3.4 The Fourier-Motzkin Elimination algorithm

Example
(i) Consider the following set of linear inequalities:

$$0 \leq x \tag{19}$$
$$-x - y \leq 2 \tag{20}$$
$$-x + y \leq 3 \tag{21}$$
$$x + 2y \leq 6 \tag{22}$$
$$0 \leq y \tag{23}$$
$$-x - y + 2 \leq z \tag{24}$$

Trasforming each of them to the x-normal form yields the following set:

$$0 \leq x \tag{25}$$
$$-y - 2 \leq x \tag{26}$$
$$y - 3 \leq x \tag{27}$$
$$x \leq -2y + 6 \tag{28}$$
$$-y \leq 0 \tag{29}$$
$$-y - z + 2 \leq x \tag{30}$$

Hence using the x-*ELIMINATION* rule we obtain the following set of inequalities in which x does not appear:

$$0 \le -2y + 6 \tag{31}$$
$$-y - 2 \le -2y + 6 \tag{32}$$
$$y - 3 \le -2y + 6 \tag{33}$$
$$-y \le 0 \tag{34}$$
$$-y - z + 2 \le -2y + 6 \tag{35}$$

Transforming each of the five inequalities to the y-normal form we now obtain the following set:

$$y \le 3 \tag{36}$$
$$y \le 8 \tag{37}$$
$$y \le 3 \tag{38}$$
$$0 \le y \tag{39}$$
$$y \le z + 4 \tag{40}$$

Eliminating now y using the y-*ELIMINATION* rule we obtain the following set of four inequalities:

$$0 \leq 3 \tag{41}$$
$$0 \leq 8 \tag{42}$$
$$0 \leq 3 \tag{43}$$
$$0 \leq z + 4 \tag{44}$$

We can now delete the first three inequalities using the *DELETION* rule and we end up with a single inequality the z-normal form which is:

$$-4 \leq z$$

At this moment we apply the z-*ELIMINATION* rule. We end up with the empty set. This implies that the original set of inequalities is consistent.

(ii) Consider now the following set of linear inequalities:

$$x + z \leq x + z + 1 \tag{45}$$
$$y + 3z + 6 \leq x + y \tag{46}$$
$$-y + 3z + 6 \leq x - y \tag{47}$$
$$x + y \leq -2y + 2 \tag{48}$$
$$x + z \leq 2y + z + 3 \tag{49}$$
$$x + 2y \leq x + z + 1 \tag{50}$$
$$x + y \leq x + z + 1 \tag{51}$$

The first inequality normalises to $-1 \leq 0$, so using the *DELETION* rule we can delete it. Transforming each of the remaining six inequalities to the x-normal form yields the following set:

$$3z + 6 \leq x \tag{52}$$

$$2z + 6 \leq x \tag{53}$$

$$x \leq -3y + 2 \tag{54}$$

$$x \leq 2y + 3 \tag{55}$$

$$2y - z - 1 \leq 0 \tag{56}$$

$$y - z - 1 \leq 0 \tag{57}$$

Using the x-*ELIMINATION* rule we obtain the following set of inequalities in which x does not appear:

$$3z + 6 \leq -3y + 2 \tag{58}$$
$$3z + 6 \leq 2y + 3 \tag{59}$$
$$2z + 6 \leq -3y + 2 \tag{60}$$
$$2z + 6 \leq 2y + 3 \tag{61}$$
$$2y - z - 1 \leq 0 \tag{62}$$
$$y - z - 1 \leq 0 \tag{63}$$

Transforming each of these six inequalities to the y-normal form we obtain the following set:

$$y \leq -z - \frac{4}{3} \tag{64}$$

$$\frac{3}{2}z + \frac{3}{2} \leq y \tag{65}$$

$$y \leq -\frac{2}{3}z - \frac{4}{3} \tag{66}$$

$$z + \frac{3}{2} \leq y \tag{67}$$

$$y \leq \frac{1}{2}z + \frac{1}{2} \tag{68}$$

$$y \leq z + 1 \tag{69}$$

So using the y-*ELIMINATION* rule we obtain a set of eight inequalities. One of them, resulting from the inequalities $z + \frac{3}{2} \leq y$ and $y \leq z + 1$ is $z + \frac{3}{2} \leq z + 1$ that normalises to $\frac{1}{2} \leq 0$. So applying the *FAILURE* rule we introduce the false constraint $\perp$ which yields a failed derivation. This means that the original set of inequalities is inconsistent.

Lemma (*INEQ*)
(i) The rules DELETION and FAILURE are equivalence preserving.
(ii) Each global application of the x-ELIMINATION rule is equivalence preserving w.r.t. the sequence of the variables present in the rule conclusion. Consequently, this rule is consistency preserving.

Theorem 4.43 (*INEQ*) *The* FOURIER-MOTZKIN ELIMINATION *algorithm always terminates. If the original finite set of linear inequalities is consistent, then each execution of the algorithm terminates with the empty set of constraints and otherwise each execution terminates with a set containing the false constraint* $\perp$.

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Exercises

Name ______________________________________-

1. Consider the following set of linear inequalities:

$$-y \leq 0 \tag{70}$$
$$-y - z + 2 \leq x \tag{71}$$
$$0 \leq x \tag{72}$$
$$-y - 2 \leq x \tag{73}$$
$$y - 3 \leq x \tag{74}$$
$$x \leq -2y + 6 \tag{75}$$

Apply once the x-*ELIMINATION* rule (to all possible inequalities in this set). Write down the resulting set of inequalities.

2. Consider the following set of linear inequalities:

$$0 \leq y \tag{76}$$
$$-x - y + 2 \leq z \tag{77}$$
$$0 \leq x \tag{78}$$
$$-x - y \leq 2 \tag{79}$$
$$-x + y \leq 3 \tag{80}$$
$$x + 2y \leq 6 \tag{81}$$

Examine by using FOURIER-MOTZKIN ELIMINATION, whether this set is consistent or not. Write down the details.