T-79.149 Discrete Structures (2 cu)

Exam Fri 10 Dec 2004, 9–12 a.m., Lecture Hall T1

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USE OF LECTURE NOTES AND SOLUTIONS TO TUTORIAL PROBLEMS PERMITTED. PROGRAMMABLE AND SYMBOLIC ALGEBRA CALCULATORS FORBIDDEN.

1. Interpret the sum

$$S_n = \binom{2n}{0} + \binom{2n-1}{1} \cdot 2 + \binom{2n-2}{2} \cdot 2^2 + \cdots + \binom{n}{n} \cdot 2^n$$

   as the coefficient of the z^{2n} term in the series $S(z) = \sum_{k \geq 0} z^k (1+2z)^k$ (justify this!), and use this interpretation to determine a closed form expression for S_n.

   8p.

2. Determine algebraic expressions for the following generating functions, based directly on the structure of the respective combinatorial families:

   (a) The ogf for sequence $\langle a_n \rangle$, where a_n = the number of strings composed of digits 1 and 2, such that the digits add up to n. (By direct counting one observes that $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$ etc.) *4p.*

   (b) The egf for sequence $\langle b_n \rangle$, where b_n = the number of "binary tree partitions" of the set $[n] = \{1, \ldots, n\}$, i.e. the number of labelled binary trees where each node contains some nonempty subset of the set $[n]$, these subsets are disjoint and together cover all of $[n]$. (A *binary tree* is an ordered rooted tree, where each nodes has two descendant subtrees, either or both of which may be empty. By direct drawing and counting one observes that $b_0 = 1$, $b_1 = 1$, $b_2 = 5$, $b_3 = 43$ etc.) *4p.*

3. Estimate, up to an order you consider appropriate, the asymptotic growth rates of the coefficient sequences of the following generating functions. Determine your estimates directly from the analytical properties of the functions, without solving the coefficient sequences explicitly.

   (a) tgf $a(z) = \dfrac{1}{(1+2z)(1-2z)^2}$. *4p.*

   (b) egf $\hat{f}(z) = z e^{z^2}$, *3p.*

4. Use Euler's summation formula to estimate the growth rate of the factorial function $n!$ in terms of n, i.e. determine some algebraic expression $f(n)$ such that $n! = \Theta(f(n))$. (*Hint:* Estimate first the function $(n-1)!$.) *7p.*

Total 30p.