## 3.8 On limitations of context-free languages

As for regular languages, a pumping lemma exists also for context-free languages. Now, however, the string must be pumped simultaneously at two locations.

**Lemma 3.9 ("The uvwxy lemma")** Let $L$ be a context-free language. Then there is an $n \geq 1$ such that any $z \in L$, $|z| \geq n$, can be split into five parts $z = uvwxy$ such that

(i) $|vx| \geq 1$,

(ii) $|vwx| \leq n$,

(iii) $uv^iwx^iy \in L$ for all $i = 0, 1, 2, \ldots$.





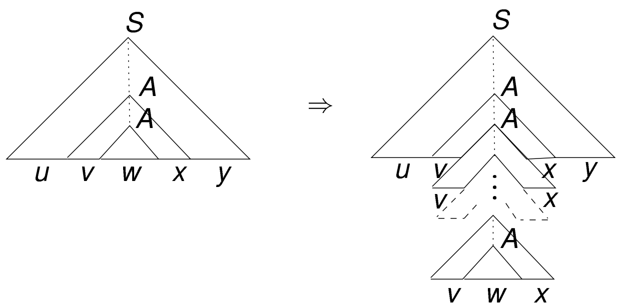
*Proof.* Let $G = (V, \Sigma, P, S)$ be a grammar for $L$ in the Chomsky normal form. Then every parse tree of $G$ with height at most $h$ has at most $2^h$ leaves. In other words, every parse tree of any $z \in L$ contains a path of length at least $\log_2 |z|$.

Let $k = |V - \Sigma|$ be the number of non-terminals in $G$. Let $n = 2^{k+1}$. Choose any $z \in L$, $|z| \geq n$, and examine its parse tree.

Based on the above, the tree contains a path of length $\geq k+1$; on this path some non-terminal must be repeated — in fact already among the $k+2$ lowest vertices. Let $A$ such a non-terminal.





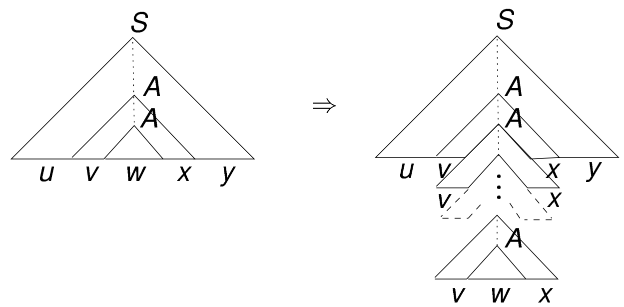



The string $z$ can now be partitioned as $z = uvwxy$, where $w$ is derived from the lowest occurrence of $A$ and $vwx$ is derived from the second lowest occurrence of $A$; the partial strings are obtained from the derivation

$$S \Rightarrow^* uAy \Rightarrow^* uvAxy \Rightarrow^* uvwxy.$$








Since $S \Rightarrow^* uAy$, $A \Rightarrow^* vAx$ and $A \Rightarrow^* w$, the partial strings $v$ and $x$ can be "pumped" around $w$:

$$S \Rightarrow^* uAy \Rightarrow^* uvAxy \Rightarrow^* uv^2Ax^2y \Rightarrow^* \ldots \Rightarrow^* uv^iAx^iy \Rightarrow^* uv^iwx^iy.$$

Thus $uv^iwx^iy \in L$ for all $i = 0, 1, 2, \ldots$.

Since $G$ is in Chomsky normal form and $A \Rightarrow^* vAx$, we must have $|vx| \geq 1$.

Further, since $A$ is chosen so that its second lowest occurrence is no higher than at height $k+1$ from the leaves of the parse tree, the length of the string derived from the second lowest occurrence of $A$ is bounded by $|vwx| \leq 2^{k+1} = n$. $\square$





**Example.** Observe the language

$$L = \{a^k b^k c^k \mid k \geq 0\}.$$

Assume that $L$ were context-free; choose the parameter $n$ according to the lemma and examine the string $z = a^n b^n c^n \in L$.

By Lemma 3.9 $z$ can be split into parts

$$z = uvwxy, \qquad |vx| \geq 1, \qquad |vwx| \leq n.$$

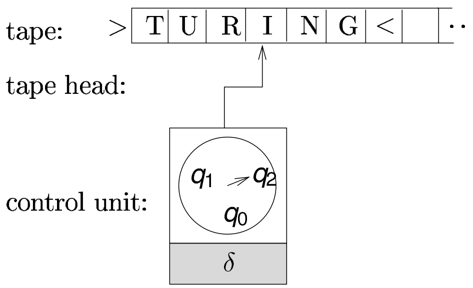
By the last condition the string $vx$ cannot contain both $a$'s, $b$'s, $c$'s. The string $uv^0wx^0y = uwy$ has an excess of some symbol, and it cannot be of the form required for belonging to $L$, even though according to the lemma we should have $uwy \in L$.





## 4. TURING MACHINES

Alan Turing 1935–36.




A Turing machine is like a finite state automaton that can store information on a tape. The machine may move the tape head left or right; it can also read or write the character at the tape head. The tape is unbounded to the right.

*The Church–Turing thesis:* Any mechanically solvable problem can be solved by a Turing machine.





Computational models equivalent to the Turing machine:

- Gödel–Kleene: recursively defined functions (1936),
- Church: $\lambda$-calculus (1936),
- Post (1936) and Markov (1951): string rewriting systems,
- all current programming languages.

Turingin machine $\equiv$ programming language.

**Definition 4.1** A *Turing machine* is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}),$$

where

- $Q$ is a finite set of *state*;
- $\Sigma$ is the *input alphabet*;
- $\Gamma \supseteq \Sigma$ is the *tape alphabet* (assume that $>, < \notin \Gamma$);
- $\delta : (Q - \{q_{acc}, q_{rej}\}) \times (\Gamma \cup \{>, <\}) \to Q \times (\Gamma \cup \{>, <\}) \times \{L, R\}$ is the *transition function*;
- $q_0 \in Q$ is the *initial state*;
- $q_{acc} \in Q$ is the *accepting* and
- $q_{rej} \in Q$ the *rejecting final state*.





Interpretation of the transition function

$$\delta(q, a) = (q', b, \Delta):$$

From state $q$ if the machine reads from the tape the character $a$, the machine moves to state $q'$, writes the character $b$ over the character it read, and moves the tape head one position to the direction $\Delta$ ($L \sim$ "left", $R \sim$ "right").

If $a$ = '$>$' or '$<$', there are limitations on the character that may be written and the direction the tape head may be moved, and the transition function is always undefined, if $q = q_{acc}$ or $q = q_{rej}$. If the machine ends up at either of these two final states, it halts immediately.





Of the values of the transition function

$$\delta(q, a) = (q', b, \Delta)$$

it is required that

(i) if $b = >$, then $a = >$;

(ii) if $a = >$, then $b = >$ and $\Delta = R$;

(iii) if $b = <$, then $a = <$ and $\Delta = L$.





The *configuration* of a Turing machine is a 4-tuple

$$(q, u, a, v) \in Q \times \Gamma^* \times (\Gamma \cup \{\varepsilon\}) \times \Gamma^*,$$

where we may have $a = \varepsilon$, if also $u = \varepsilon$ or $v = \varepsilon$.

Interpretation: the machine is in state $q$, the contents of the tape from the beginning up to but not including the tape head is $u$, at the tape head is the character $a$, and to the right of the tape head to the end of the used section of tape there is the string $v$.

We may have $a = \varepsilon$, if the tape head is at the beginning or end of the used part of the tape. In the first case, the machine is thought to "see" the character '$>$' and in the second case the character '$<$'.

The *initial configuration with input* $x = a_1 a_2 \dots a_n$ is the 4-tuple

$$(q_0, \varepsilon, a_1, a_2 \dots a_n).$$

The configuration $(q, u, a, v)$ is often denoted more simply by $(q, u\underline{a}v)$, and the initial configuration with input $x$ simply by $(q_0, \underline{x})$.

The configuration $(q, w)$ *leads directly* to configuration $(q', w')$, denoted by

$$(q, w) \underset{M}{\vdash} (q', w'),$$

if one of the following is true: for all $q, q' \in Q$, $u, v \in \Gamma^*$, $a, b \in \Gamma$ ja $c \in \Gamma \cup \{\varepsilon\}$:

if $\delta(q, a) = (q', b, R)$, then $(q, u\underline{a}cv) \underset{M}{\vdash} (q', ub\underline{c}v)$;

if $\delta(q, a) = (q', b, L)$, then $(q, uc\underline{a}v) \underset{M}{\vdash} (q', u\underline{c}bv)$;

if $\delta(q, >) = (q', >, R)$, then $(q, \underline{\varepsilon}cv) \underset{M}{\vdash} (q', \underline{c}v)$;

if $\delta(q, <) = (q', b, R)$, then $(q, u\underline{\varepsilon}) \underset{M}{\vdash} (q', ub\underline{\varepsilon})$;

if $\delta(q, <) = (q', b, L)$, then $(q, uc\underline{\varepsilon}) \underset{M}{\vdash} (q', u\underline{c}b)$;

if $\delta(q, <) = (q', <, L)$, then $(q, uc\underline{\varepsilon}) \underset{M}{\vdash} (q', u\underline{c})$.

Configurations of the form $(q_{acc}, w)$ or $(q_{rej}, w)$ lead to no other configuration. In such configurations the machine *halts*.





The configuration $(q, w)$ *leads* to configuration $(q', w')$, denoted by

$$(q, w) \underset{M}{\vdash}^* (q', w'),$$

if there exists a sequence of configurations $(q_0, w_0), (q_1, w_1), \ldots, (q_n, w_n)$, $n \geq 0$, such that

$$(q, w) = (q_0, w_0) \underset{M}{\vdash} (q_1, w_1) \underset{M}{\vdash} \cdots \underset{M}{\vdash} (q_n, w_n) = (q', w').$$

A Turing machine $M$ *accepts* the string $x \in \Sigma^*$, if

$$(q_0, \underline{x}) \underset{M}{\vdash}^* (q_{acc}, w) \qquad \text{for some } w \in \Gamma^*;$$

otherwise $M$ *rejects* $x$.

The language $M$ *recognizes* is

$$L(M) = \{x \in \Sigma^* \mid (q_0, \underline{x}) \underset{M}{\vdash}^* (q_{acc}, w) \text{ for some } w \in \Gamma^*\}.$$





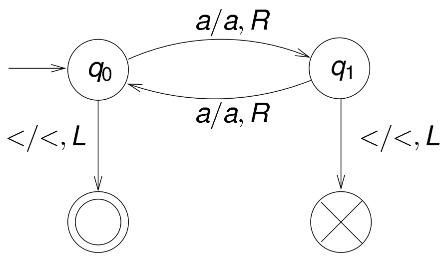
**Example 1.** The language $\{a^{2k} \mid k \geq 0\}$ can be recognized by the Turing machine

$$M = (\{q_0, q_1, q_{acc}, q_{rej}\}, \{a\}, \{a\}, \delta, q_0, q_{acc}, q_{rej}),$$

missä

$$\begin{aligned} \delta(q_0, a) &= (q_1, a, R), \\ \delta(q_1, a) &= (q_0, a, R), \\ \delta(q_0, <) &= (q_{acc}, <, L), \\ \delta(q_1, <) &= (q_{rej}, <, L). \end{aligned}$$

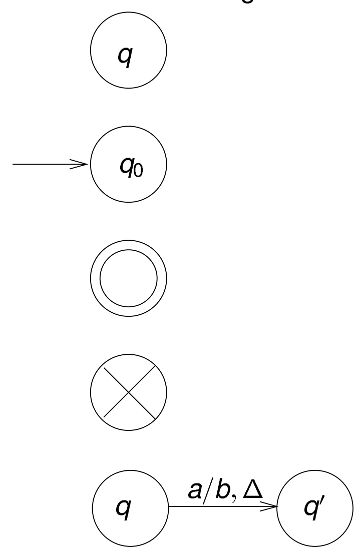
A diagram presentation:








Notation used in diagram:



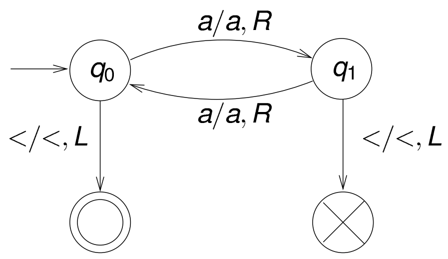

State $q$

Initial state

Accepting final state $(q_{acc})$

Rejecting final state $(q_{rej})$

State transition $\delta(q, a) = (q', b, \Delta)$




The computation of $M$ with for example input $aaa$ proceeds as follows:

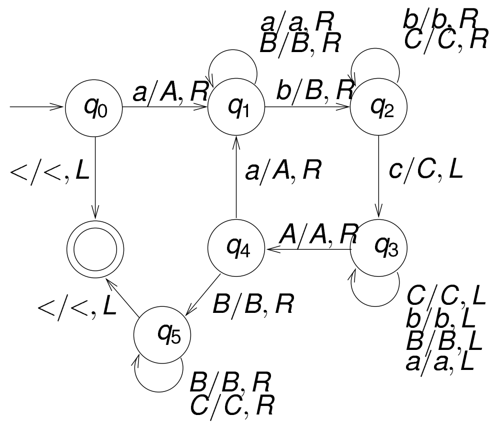
$$(q_0, \underline{a}aa) \underset{M}{\vdash} (q_1, a\underline{a}a) \underset{M}{\vdash} (q_0, aa\underline{a}) \underset{M}{\vdash} (q_1, aaa\underline{\varepsilon}) \underset{M}{\vdash} (q_{rej}, aa\underline{a}).$$

The machine halts in state $q_{rej}$, so $aaa \notin L(M)$.





**Example 2.** A machine that recognizes the language $\{a^k b^k c^k \mid k \geq 0\}$:




For clarity the rejecting final state $q_{rej}$ is not shown explicitly. Then the interpretation is that all "missing" edges lead to this state.





The computation with input $aabbcc$:

$(q_0, \underline{a}abbcc) \vdash (q_1, A\underline{a}bbcc) \vdash$
$(q_1, Aa\underline{b}bcc) \vdash (q_2, AaB\underline{b}cc) \vdash$
$(q_2, AaBb\underline{c}c) \vdash (q_3, AaB\underline{b}Cc) \vdash$
$(q_3, Aa\underline{B}bCc) \vdash (q_3, A\underline{a}BbCc) \vdash$
$(q_3, \underline{A}aBbCc) \vdash (q_4, A\underline{a}BbCc) \vdash$
$(q_1, AA\underline{B}bCc) \vdash (q_1, AAB\underline{b}Cc) \vdash$
$(q_2, AABB\underline{C}c) \vdash (q_2, AABBC\underline{c}) \vdash$
$(q_3, AABB\underline{C}C) \vdash (q_3, AAB\underline{B}CC) \vdash$
$(q_3, AA\underline{B}BCC) \vdash (q_3, A\underline{A}BBCC) \vdash$
$(q_4, AA\underline{B}BCC) \vdash (q_5, AAB\underline{B}CC) \vdash$
$(q_5, AABB\underline{C}C) \vdash (q_5, AABBC\underline{C}) \vdash$
$(q_5, AABBCC\underline{\varepsilon}) \vdash (q_{acc}, AABBC\underline{C})$.