T-79.148 **Spring 2003**
Introduction to Theoretical Computer Science
Tutorial 2, 27–29 January
Problems

Homework problems:

1. Let $\Sigma = \{a, b\}$. Give some examples of strings from each of the following languages (at least three strings per language):
   (a) $\{w \in \Sigma^* \mid w$ the number of a's in w is even and the number of b's is divisible of three$\}$;
   (b) $\{a^{2n}b^{3m} \mid n, m \geq 0\}$;
   (c) $\{uvu^Rv^R \mid u, v \in \Sigma^*\}$;[1]
   (d) $\{w \in \Sigma^* \mid \exists u, v \in \Sigma^* \text{ s.t. } w = uu = vvv\}$.

2. The *reversal* of a string $w \in \Sigma^*$, denoted w^R, is defined inductively by the rules:
   (i) $\varepsilon^R = \varepsilon$;
   (ii) if $w = ua$, where $u \in \Sigma^*$ and $a \in \Sigma$, then $w^R = au^R$.

   It was proved in class (cf. also Lewis & Papadimitriou, p. 43) that for any strings $u, v \in \Sigma^*$ it is the case that $(uv)^R = v^Ru^R$. Prove in a similar manner, by induction based on the above definition of reversal, the following facts:
   (a) $(w^R)^R = w$;
   (b) $(w^k)^R = (w^R)^k$, for any $k \geq 0$.

3. Prove that the union of two countably infinite sets is countably infinite. Deduce from this by induction that the same holds for the union of n countably infinite sets, for any $n = 1, 2, \ldots$ (*Extra question:* Does the claim still hold if the number of sets to be combined is countably infinite, e.g. $A = A_1 \cup A_2 \cup \ldots$, where each A_i is countably infinite?)

Demonstration problems:

4. Show that any alphabet Σ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over Σ can be easily encoded into strings over Γ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only *one* symbol, e.g. $\Gamma = \{1\}$?

5. Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean (x, y) plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

6. Let S be an arbitrary nonempty set.
   (a) Give some injective (i.e. one-to-one) function $f : S \to \mathcal{P}(S)$.
   (b) Prove that there cannot exist an injective function $g : \mathcal{P}(S) \to S$. (*Hint:* Assume that such a function g existed. Consider the set $R = \{s \in S \mid s \notin g^{-1}(s)\}$, and denote $r = g(R)$. Is it then the case that $r \in R$?)

   Observe, as a consequence of item (b), that the power set $\mathcal{P}(S)$ of any countably infinite set S is uncountable.

[1] For a definition of the notation w^R see Problem 2.