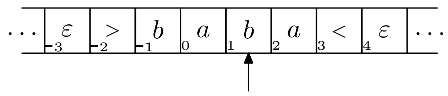
T-79.148 **Autumn 2003**
Introduction to Theoretical Computer Science
Tutorial 10
Solutions to the demonstration problems

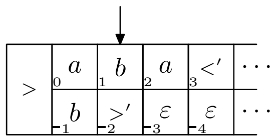
4. **Problem**: Extend the notion of a Turing machine by providing the possibility of a two-way infinite tape. Show that nevertheless such machines recognize exactly the same languages as the standard machines whose tape is only one-way infinite.

   Solution: A Turing machine with a two-way infinite tape works otherwise in a same way than a standard machine except that the position of the tape start symbol ($>$) is not fixed and it can move in a same way than the end symbol ($<$). The tape positions are indexed by the set $\mathbb{Z}$ of integers where 0 denotes the initial position of $>$.

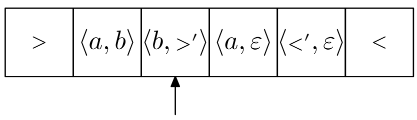
   We can simulate such a Turing machine by a two-track one-way Turing machine. Conceptually, we divide the tape into two parts: upper and lower. The upper part holds the two-way tape cells $i \geq 0$ and the lower part cells $i < 0$. For example, a two-way tape:



   is expressed as a one-way tape:



   In practice the construction of two tracks is done by replacing the alphabet Σ by a new alphabet $\Sigma' = (\Sigma \cup \{<', >'\}) \times (\Sigma \cup \{<', >'\})$. Each symbol of Σ' thus denotes two symbols of Σ. The symbols $\{<', >'\}$ are new symbols that denote the start and end symbols of the original tape. So, the above example is expressed as:



   We still need a way to decide which tape-half is used. The easiest way to do this is to define a mirror image state q' for each state q. When the machine is in state q, it examines only the upper track when it decides what move to take next (tape head is on right side of the tape). Similarily, in state q' it examines only the lower symbol (tape head is on the left side). Since the lower tape is in a reversed order, all tape head moving instructions have to be also reversed.

   The formal definition of this construction is presented in an appendix.

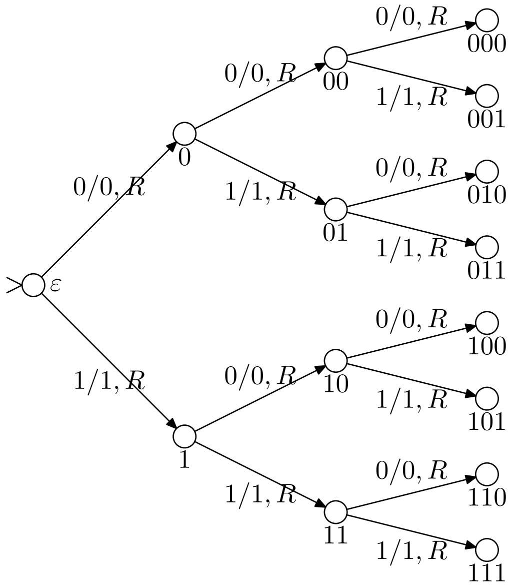
5. **Problem**: Show that Turing machines whose tape alphabet contains at most two symbols in addition to the input symbols are capable of recognising exactly the same languages as the standard machines.

   Solution:

   Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$ be a Turing machine such that $|\Gamma - \Sigma| > 2$. We want to construct a machine M' such that $\Gamma' = \{0, 1\}$. Let $\Gamma = \{a_1, \ldots, a_n\}$. The basic idea of the construction is to identify the elements of Γ with the integers $\{\ 1, \ldots, n\ \}$ and represent them as k-bit integers, where $k = \lceil \log_2(|\Gamma|)$. In other words, each element of M's tape alphabet is replaced with k bits. For example, suppose that $N = 3$ and the tape has the input $a_1a_2a_3$. In this case the encoding is:

$$\boxed{>}\boxed{\underline{a_1}}\boxed{a_2}\boxed{a_3}\boxed{<} \Longrightarrow \boxed{>}\boxed{\underline{0}}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{<}$$

The transition function of M' is defined so that for each step of M, M' does first k steps where it first decides what symbol of Γ is encoded in the tape cells to the right of the read/write head. This can be done using a Turing machine that reads k symbols from the tape while moving its head to right at each step and that remembers the input in its states. For example, if $k = 3$, then the following Turing machine may be used:



If the machine ends in the state 011, then the input symbol is a_3 since $011_2 = 3_{10}$. The symbol that is written to the tape is similarily done using k different transitions. Finally, the tape head is moved k steps to the correct direction.

Appendix: the formalisation of solution 4

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$ be a two-way tape Turing machine. Define a standard Turing machine M' as follows:

$$\begin{aligned}
M' &= (Q', \Sigma', \Gamma', \delta', q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}}) \\
Q' &= Q \cup \{q' \mid q \in Q\} \\
\Sigma' &= (\Sigma \cup \{<', >'\}) \times (\Sigma \cup \{<', >'\}) \\
\Gamma' &= (\Gamma \cup \{<', >'\}) \times (\Gamma \cup \{<', >'\})
\end{aligned}$$

The transition function δ' is defined as follows:

$$\begin{aligned}
\delta' = \{&(q_1, \langle a, \gamma\rangle, q_2, \langle b, \gamma\rangle, \Delta) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma'\} \\
\cup \{&(q_1, \langle \sigma', \gamma\rangle, q_2, \langle b, \gamma\rangle, \Delta) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\
\cup \{&(q_1', \langle \gamma, a\rangle, q_2', \langle \gamma, b\rangle, \overline{\Delta}) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma'\} \\
\cup \{&(q', \langle \gamma, a\rangle, q_{\mathrm{end}}, \langle \gamma, b\rangle, \overline{\Delta}) \mid (q, a, q_{\mathrm{end}}, b, \Delta) \in \delta, q_{\mathrm{end}} \in \{q_{\mathrm{acc}}, q_{\mathrm{rej}}\}, \gamma \in \Gamma'\} \\
\cup \{&(q_1', \langle \gamma, \overline{\sigma}'\rangle, q_2', \langle \gamma, b\rangle, \overline{\Delta}) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\
\cup \{&(q, >, q', >, R), (q', >, q, >, R) \mid q \in Q\},
\end{aligned}$$

where $\overline{L} = R$, $\overline{R} = L$, $\overline{<} = >$ and $\overline{>} = <$.