T-79.1001 **Kevät 2006**
Introduction to Theoretical Computer Science (T)
Session 11
Answers to demonstration exercises

4. **Problem**:

   Prove, without appealing to Rice's theorem, that the following problem is undecidable:

   Given a Turing machine M; does M accept the empty string?

   Solution:

   First we define a language $L = \{M \mid M \text{ halts with the input } \varepsilon\}$. Now, L is recursive if and only if the decision problem in the exercise statement is decisive. Next we show that the language $H = \{Mw \mid M \text{ halts with input } w\}$ can be recursively reduced to L (denoted $H \leq_m L$) so L is at least as difficult as H. Since H is not recursive, L may not be recursive, either.

   The concept of a recursive reduction is defined as follows: Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages. Now $A \leq_m B$ if and only if there exists a recursive function$f : \Sigma^* \to \Gamma^*$ such that
   $$\forall w \in \Sigma^* : w \in A \Leftrightarrow f(w) \in B \ .$$

   In this case we want to find a function f such that $f(Mw) \in L$ if and only if $Mw \in H$. In practice this means that we want to find a systematic way to construct a Turing machine M' that halts with an empty input exactly when M halts with $w = w_1 w_2 \cdots w_n$.

   Fortunately, this is an easy thing to do: M' starts by writing w to its tape and after that it simulates M. Now M' stops only if M stops.

   Formally, f can be defined as:

   $$f(\langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}\rangle, w_1 w_2 \cdots w_n) = \langle Q', \Sigma, \Gamma, \delta', q_0', q_{\text{acc}}, q_{\text{rej}}\rangle,$$

   where

   $$\begin{aligned} Q' &= Q \cup \{q_i' \mid 0 \leq i \leq n\} \\ \delta' &= \delta \cup \{\langle q_i', \varepsilon, q_{i+1}', w_{i+1}, R\rangle \mid 0 \leq i < n\} \\ &\quad \cup \{\langle q_n', x, q_n', x, L\rangle \mid x \in \Gamma \cup \{<\}\} \\ &\quad \cup \{\langle q_n', >, q_0, >, R\rangle\} \end{aligned}$$

   Since we add only a finite number of states and transitions to M (n has to be finite), f is trivially recursive.

5. **Problem**: Prove the following connections between recursive functions and languages:

   (i) A language $A \subseteq \Sigma^*$ is recursive ("Turing-decidable"), if and only its characteristic function
   $$\chi_A : \Sigma^* \to \{0,1\}, \qquad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases}$$
   is a recursive ("Turing-computable") function.

   (ii) A language $A \subseteq \Sigma^*$ is recursively enumerable ("semidecidable", "Turing-recognisable"), if and only if either $A = \emptyset$ or there exists a recursive function $g : \{0,1\}^* \to \Sigma^*$ such that
   $$A = \{g(x) \mid x \in \{0,1\}^*\}.$$

Solution: We start by defining five simple helper machines:

- **1** writes '1' to the input tape, moves the read/write head to right and stops.
- **0** writes '0' to the tape and stops.
- C empties the input tape, moves the head to the beginning of the tape and stops.
- NEXT reads the input $x \in \Sigma^*$ and replaces it with the lexicographic successor of x.
- $Cmp^{i,j}$ compares the contents of the input tapes i and j of a multi-tape Turing machine and accepts if they are identical.

Since the machines are simple, they are not presented here.

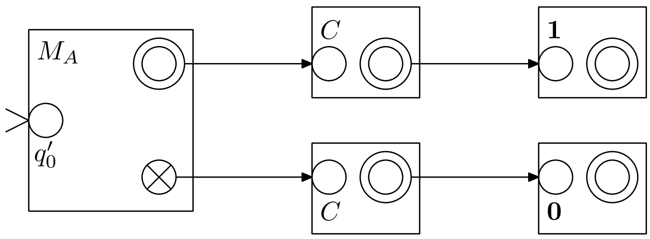
(i) [$\Rightarrow$] Let $A \subseteq \Sigma^*$ be a recursive language. Then there exists a Turing machine M_A:

$$M_A = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}} \rangle$$

such that

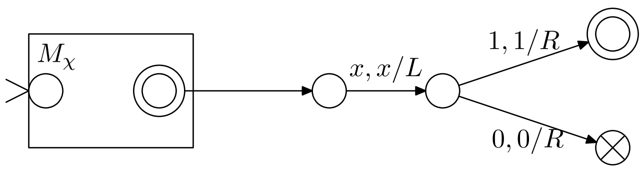
$$\begin{aligned} \forall w \in \Sigma^* : w \in L \Leftrightarrow (q_0, w) \vdash^*_{M_A} (q_{\mathrm{acc}}, \alpha) \quad &\text{ja} \\ w \notin L \Leftrightarrow (q_0, w) \vdash^*_{M_A} (q_{\mathrm{rej}}, \alpha) \end{aligned}$$

We construct a machine M by combining M_A with machines **1**, **0**, C as follows:



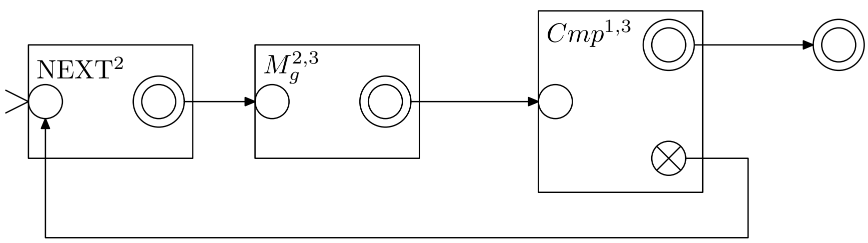
If $w \in L$, then M_A accepts w. After that M clears the tape and writes 1 to the tape. Otherwise 0 is written. Since A is recursive, M_A halts always so also M halts and it computes the function $\chi(w) = \begin{cases} 1, w \in A \\ 0, w \notin A \end{cases}$ that is the characteristic function of A.

[$\Leftarrow$] Suppose that the function $\chi(w)$ is recursive. Then there exists a Turing machine M_χ that computes it. We can now construct a machine M as follows:



Now M accepts w whenever $\chi(w) = 1$ and rejects it when $\chi(w) = 0$, so M decides the language A and A is recursive.

(ii) If $A = \emptyset$, then trivially $A \in RE$ and $g(x) = 0$ is its characteristic function.
If there exists a function g that fulfills the conditions, then there exists a Turing machine M_g that computes g. We can trivially modify it so that it becomes a 2-tape machine $M_g^{1,2}$ that computes g but stores the result in the second tape instead of the first. We now construct a 3-tape machine as follows:



The machine gets its input from its first tape and it stays untouched for the whole computation. In each iteration M_A replaces the bit string x on the second tape by its lexicographic successor y, computes $g(y)$ and writes the output on the third tape. Finally, the contents of tapes 1 and 3 are compared and if they match, the word is accepted, otherwise the iteration proceeds into the next round.

[$\Leftarrow$] Consider the word $w \in A$. Suppose that a recursive function g that fulfills the conditions exists. Then $w = g(x)$ for some $x = x_1 x_2 \cdots x_n$ where n is finite. Since each finite string has a finite number of predecessors in the lexicographic order, NEXT eventually generates x, $M_g^{2,3}$ generates w on the third tape and M_A accepts the word. Thus, M_A recognizes the language A so $A \in RE$.

[$\Rightarrow$] Next, suppose that $A \in RE - \{\emptyset\}$. Then there exists a Turing machine M_A that recognizes it. We now define a helper machine $M_{A,i}$ that simulates M_A for i steps. The machine $M_{A,i}$ accepts x if M_A accepts it using at most i steps, and rejects it otherwise. We note that $M_{A,i}$ always halts.

We construct the function g with the help of $M_{A,i}$. Every input x and bound i is encoded into bit strings using the function $c(x, y) = 0^x 1 0^y$. We define that $g(c(x, y)) = x$, if $M_{A,y}$ accepts x. We define that $g' : \{0,1\}^* \to \{0,1\}^*$ is the function:

$$g'(w) = \begin{cases} x, & w = 0^x 1 0^y \text{ and } M_{A,y}(x) \text{ accepts} \\ x_0, & \text{otherwise} , \end{cases}$$

where $x_0 \in A$. Finally, $g(x) = d(g'(x))$ where d is a function that maps a bit string 0^x into the xth element of n Σ^* in the lexicographic order. The value of g' may be computed in a finite time since $M_{A,y}(x)$ always halts. Thus, g' is recursive and so also g is.

Note that while g always exists, it is not always possible to find it since in the general case it is an undecidable problem to find an element $x_0 \in A$ that is needed for the definition.