T-79.7003 Research Course in Theoretical Computer Science:
Phase Transitions in Optimisation Problems

Combinatorial Auctions

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Combinatorial auctions (CAs)

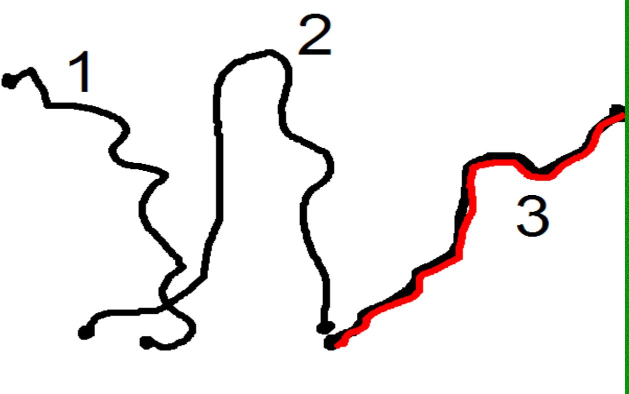
- Auctioneer has many items for sale
- Bidders bid on combinations of items
- Why not traditional single-item single-bid?
  - Higher economic efficiency
  - Higher auctioneer revenue

Combinatorial Auctions, ed. by Cramton, Shoham, Steinberg (2006)

Allocating bus routes with a combinatorial auction

- 3 companies, 3 routes

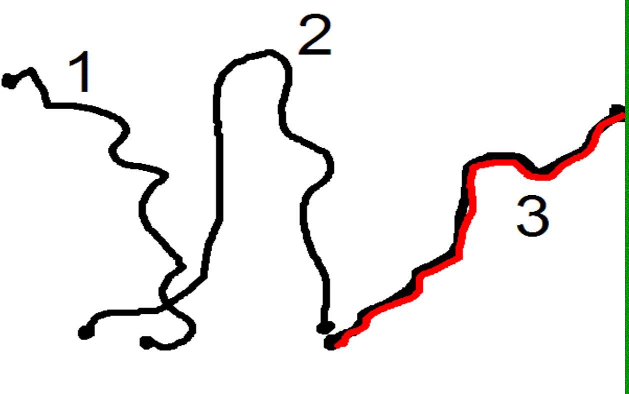
	1	2	3	1+2	2+3	1+3	1+2+3
Lucky							
Mega							
Safe							



Allocating bus routes with a combinatorial auction

Bids (M€)

	1	2	3	1+2	2+3	1+3	1+2+3
Lucky	1	1	0	2	0	0	0
Mega	0	0	5	5	7	5	10
Safe	4	0	0	0	0	0	8



- Traditional: revenue 10 M€
- Combinatorial: revenue 11 M€

For example…

- Allocation of airport time slots for takeoff and landing
- Truckload transportation
- Allocation of bus routes
- Industrial procurement
- Allocation of radio spectrum licenses

Winner determination problem (WDP)

- Given a set of bids, find an allocation of items to bidders such that the auctioneer's revenue (or number of satisfied bidders) is maximized
- NP-complete combinatorial optimization problem

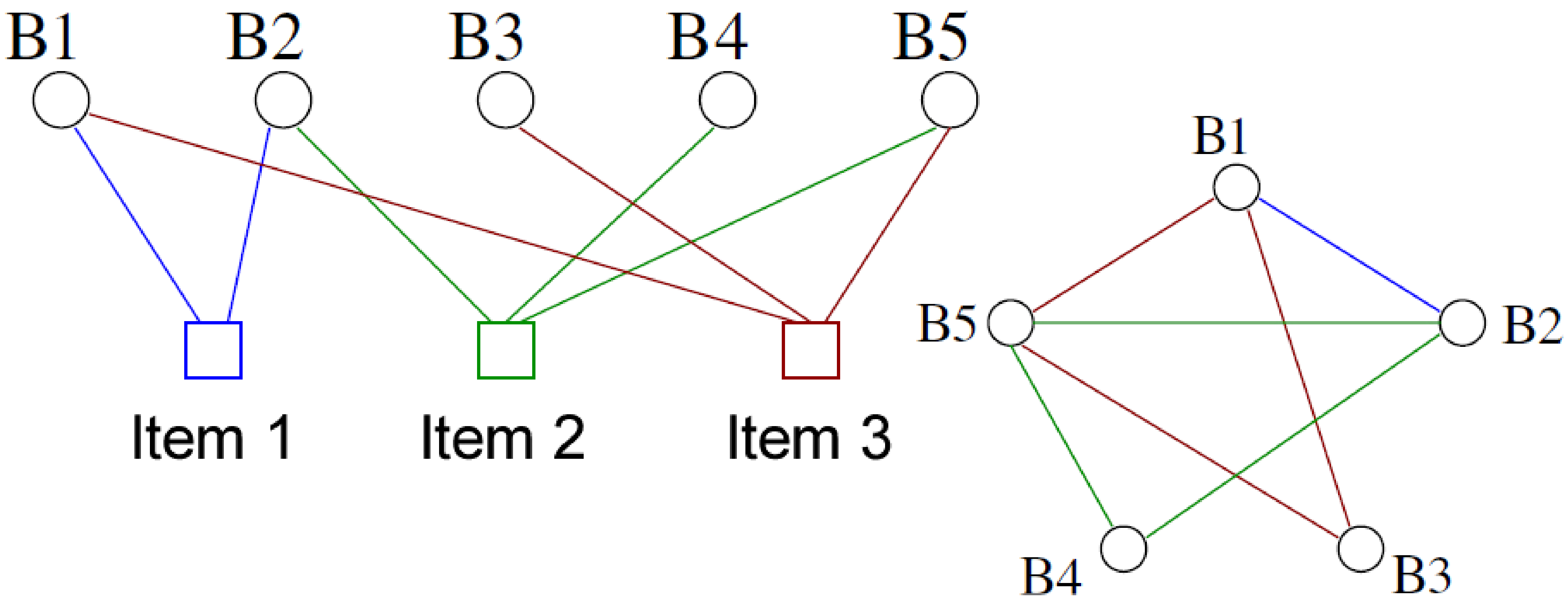
Statistical mechanics model: Definitions (1)

- N bidders, M items to be sold
- Each bidder i bids price v_i on one set of items $A_i \Rightarrow$ a bid = (A_i, v_i)
- Multiple bids nested by ORs or XORs reduces to the above
- $x_i = 1$: bidder i wins
- $x_i = 0$: bidder i loses

Statistical mechanics model: Definitions (2)

- WDP: Find a configuration $\mathbf{x} = (x_1,\ldots,x_N) \in \{0,1\}^N$ which
  1. maximizes revenue R, or
  2. number of satisfied bidder N_s, or
  3. both
- Each item sold at most once
- Bids A_i can overlap

Factor graph & conflict graph

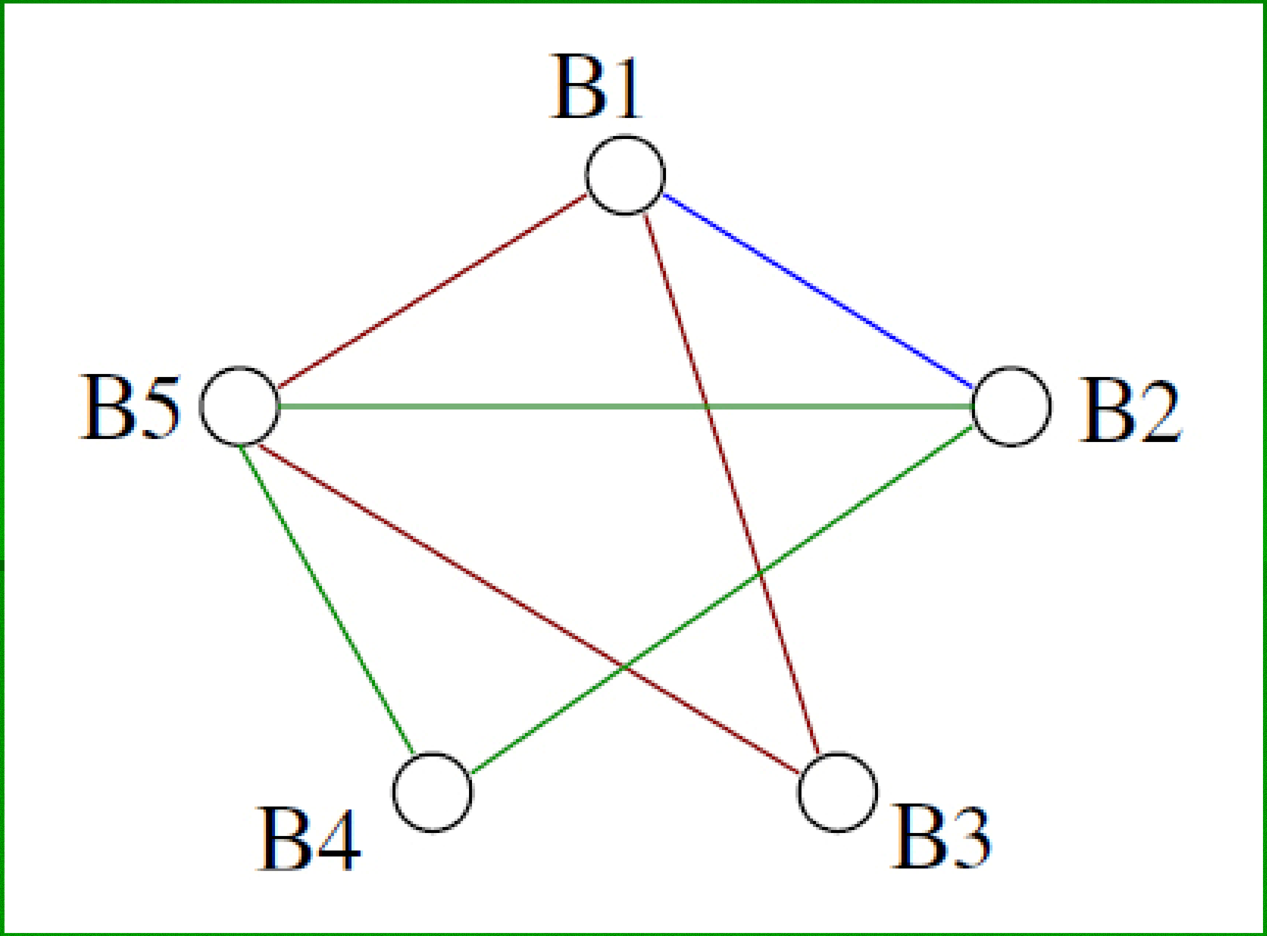


From Galla *et al.*, PRL **97**, 128701 (2006)

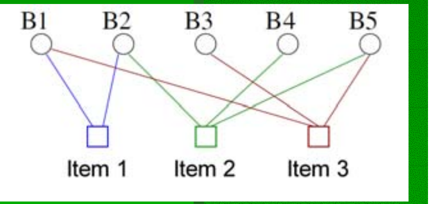
Similarity to the vertex cover problem

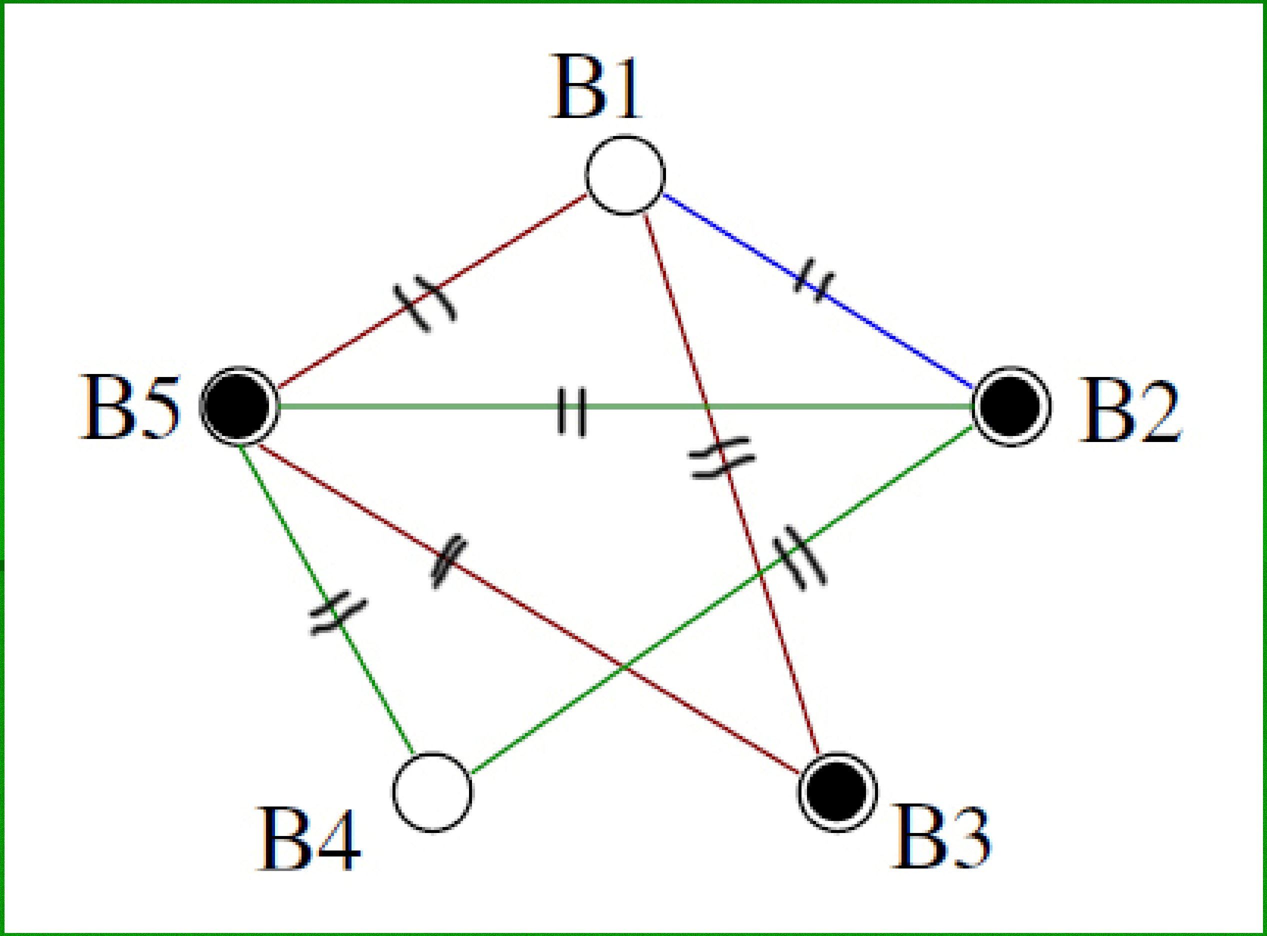
- Allocate the items to the bidders ⇔ Find a vertex cover for the CG
  - Uncovered bids win
  - Edge covered ⇔ Conflict removed
- Winner determination problem = Weighted vertex cover problem
- Maximize N_s ⇔ Find a minimum VC

Vertex cover for the conflict graph



Vertex cover for the conflict graph





Physics

- CA ~ lattice gas on the CG
- $H = -R = -\sum_{i=1}^{N} \nu_i x_i$
- $N_s = \sum_i x_i$
- $Z_\mu = \sum_{\mathbf{x}} \left[\exp\left(\sum_{i=1}^{N} (\mu + \beta \nu_i) x_i \right) \times \prod_{(i,j)\in \mathrm{CG}} (1 - x_i x_j) \right]$

Cavity-method algorithm

- Search for the optimal configuration in combinatorial auction instances
- Cavity bias $u_{a \to i}$ = likelihood that item *a* is already assigned to another bid
- Cavity field $h_{i \to a}$ = likelihood that *i* would win if A_i did not contain item *a*
- Self-consistent eqs. for $u_{a \to i}$ and $h_{i \to a}$

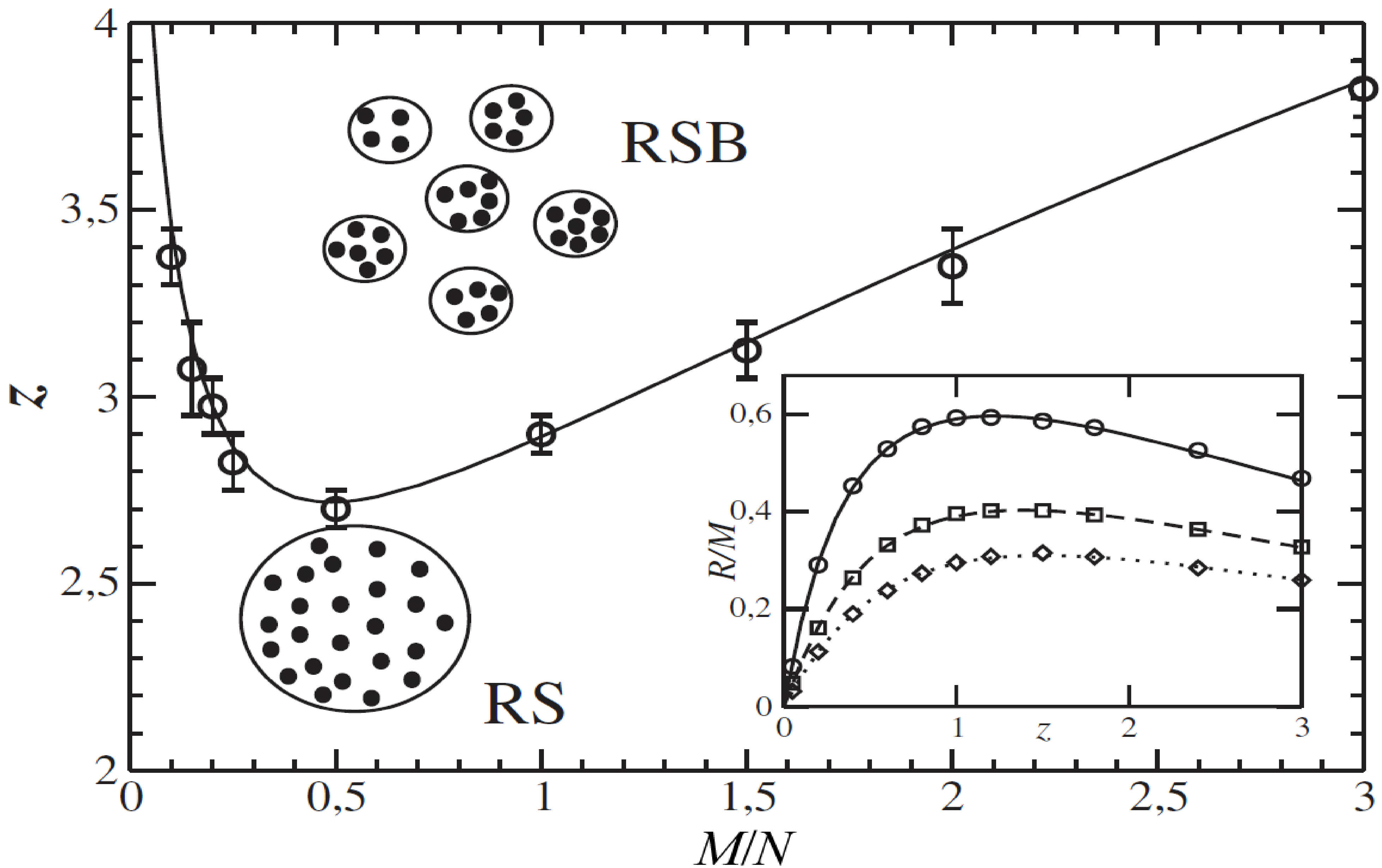
Clearing the auction

- Belief propagation
- Solve by iteration ⇒ local field H_i
- If auctioneer maximizes revenue
  - H_i = gap between bid and winning price
  - $H_i > 0$ ⇒ bid wins; $H_i < 0$ ⇒ bid loses
  - Assign items to highest H_i and start over
- Entire procedure in $O(N \log N)$ steps

Typical behaviour

- Average over random factor-graph and price ensembles
- Self-consistent integral equations
- Each bidders selects each item with probability z/M
  - Independently

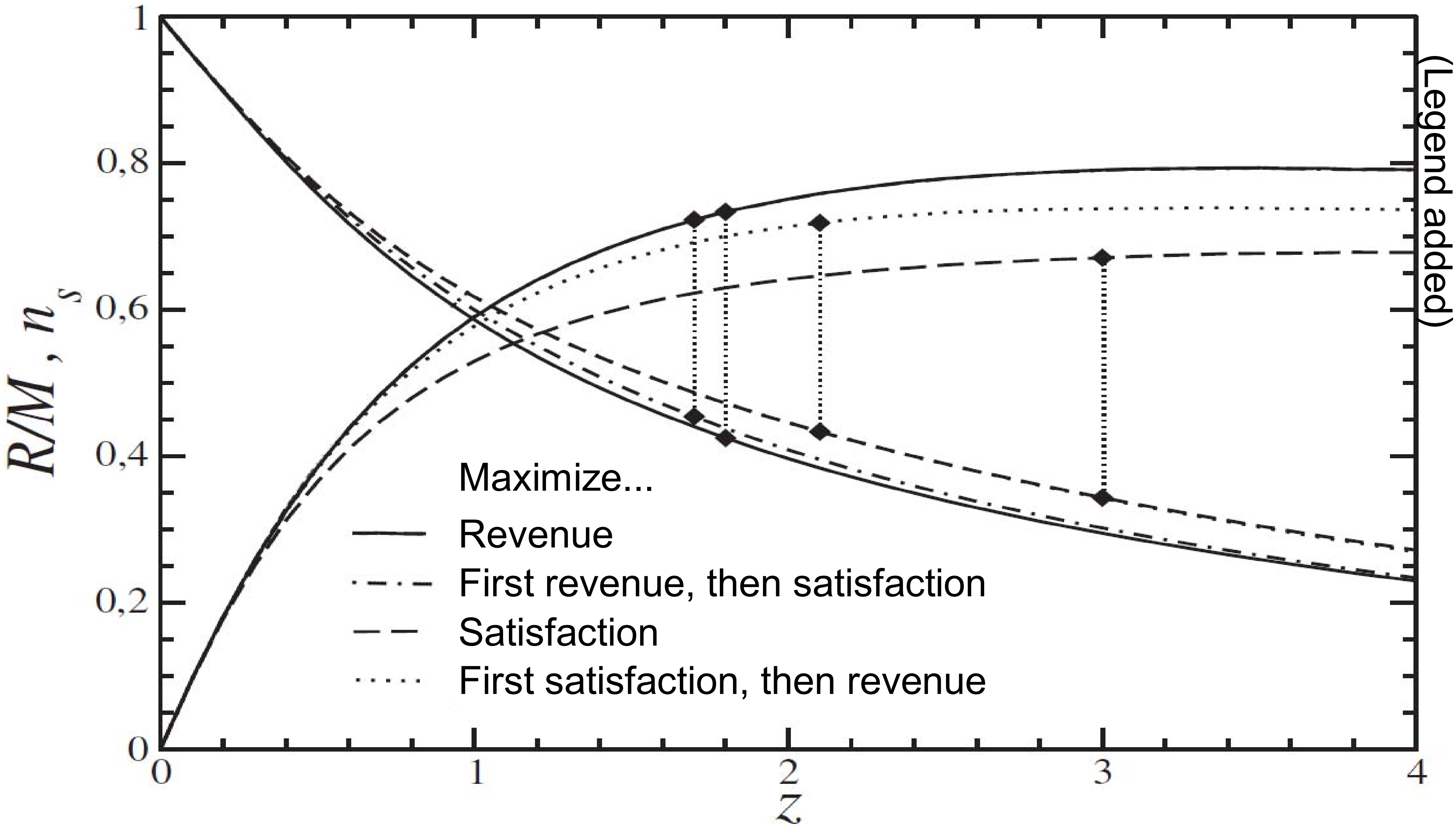
Constant prices ($v_i = 1$)



Galla *et al.*, PRL **97**, 128701 (2006)

Next: Summary!

Linear prices ($v_i \sim |A_i|$)



Galla *et al.*, PRL **97**, 128701 (2006)

Summary

- Combinatorial auctions are reality
- WDP is a hard combinatorial optimization problem
- Replica symmetry breaking in the WDP