Vertex cover on other graph ensembles: the effect of degree–degree correlations

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- Introduction
- Generalized correlated random graphs (GCRG’s)
- Lattice gas on GCRG
- Numerics
- Applications
- Conclusions
Introduction

Generalized correlated random graphs (GCRG’s)

Lattice gas on GCRG

Numerics

Applications

Conclusions
Introduction

• During the course we’ve seen VC’s on Erdös-Renyi random graphs

• Real-worlds graphs are more complicated
  – Non-poissonian degree distributions (often fat tails)
  – Degree–degree correlations

• VC’s on such graphs are important since they have applications in, for instance, network traffic monitoring
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(a) Random Network  (b) Scale-free network

N=32  m=33
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GCRG’s

- A set on undirected graphs with $N$ vertices and an arbitrary degree distribution $p_d$
- An important quantity is the excess degree distribution

$$q_d = \frac{(d + 1)p_{d+1}}{\langle d \rangle}$$  \hspace{1cm} (1)
GCRG’s (cont’d)

• Choose an edge randomly: the endpoints have excess degrees $d$ and $d'$ with probability

$$(2 - \delta_{d,d'})e_{dd'}$$

(2)

• $e_{dd'}$ is related to the conditional probability that a vertex of degree $d$ is reached coming from a vertex of degree $d'$

$$P(d|d') = \frac{e_{dd'}}{q_{d'}}$$

(3)

• For uncorrelated graphs $e_{dd'} = q_d q_{d'}$
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Lattice gas

• Consider an arbitrary undirected graph with adjacency matrix $J_{ij}$

• A general lattice gas on the graph is defined by the Hamiltonian

$$-\beta H = \sum_{i<j} J_{ij} w(x_i, x_j) + \mu \sum_i x_i$$  \hspace{1cm} (4)

• The microscopic degrees of freedom $x_i = 0, 1$

• $\mu$ is the chemical potential

• The ferromagnetic Ising model is recovered by choosing

$$w(x_i, x_j) = (2x_i - 1)(2x_j - 1)$$
Lattice gas (cont’d)

• Vázquez and Weigt perform a cavity calculation of the system
• In the end, it will be applied the VC’s to estimate the relative size of the minimum vertex cover
• The calculation touches the issue of replica symmetry breaking without actually being an RSB calculation
• In this talk, I will next go through the main parts of the calculation
Partition functions

- Usually the partition function reads

\[ Z = \sum_{\text{all states}} e^{-\beta H} \quad (5) \]

- Assume the graph is locally treelike

- Now consider an arbitrary edge \((i, j)\) and the subtree rooted in \(i\) with edge \((i, j)\) removed
Partition functions (cont’d)

• Write down the partition functions with $x_i$ fixed to value $x$

\[ Z_0^{(i|j)} = \prod_{k \neq j, J_{ik} = 1} \left( e^{w(0,0)Z_0^{(k|i)}} + e^{w(0,1)Z_1^{(k|i)}} \right) \]

\[ Z_1^{(i|j)} = e^\mu \prod_{k \neq j, J_{ik} = 1} \left( e^{w(1,0)Z_0^{(k|i)}} + e^{w(1,1)Z_1^{(k|i)}} \right) \]
Effective fields

• Define the effective fields as

\[ h_{(i|j)} = \ln \frac{Z_{1}^{(i|j)}}{Z_{0}^{(i|j)}} \]  \hspace{1cm} (6)

• Physical meaning: an isolated particle with \(-\beta H = h x, Z_{1} = e^{h}, Z_{0} = 1\) and \(h = \ln(Z_{1}/Z_{0})\)

• Using Eqs. from the previous slide we get

\[ h_{(i|j)} = \mu + \sum_{k \neq j \mid J_{ik} = 1} u(h_{k|i}) \]  \hspace{1cm} (7)

\[ u(h_{k|i}) = \ln\left(\frac{e^{w(1,0) + h_{(k|i)}} + e^{w(1,1) + h_{(k|i)}}}{e^{w(0,0)} + e^{w(0,1) + h_{(k|i)}}}\right) \]  \hspace{1cm} (8)
Iteration and RS

- The equation for $h_{i|j}$ on the previous slide defines an iteration: for each step the $h$'s can be substituted on the right-hand side, and new $h$'s obtained.

- The assumption that this iteration converges to a well-defined probability distribution $P(h)$ of the $h$'s corresponds to the assumption that the replica symmetry is not broken.

- $P_d(h)$ for nodes of degree $d$ is given by

\[
P_d(h) = \int_{-\infty}^{\infty} \prod_{l=1}^{d} (dh_l) \sum_{d'=0}^{\infty} p(d'|d) P_{d'}(h_l)) \delta(h - \mu - \sum_{l=1}^{d} u(h_l)) \quad (9)
\]
Back to vertex covers

• The lattice gas is a vertex cover with the choice

\[ e^{w(x_i, x_j)} = 1 - x_i x_j \]  \hspace{1cm} (10)

• Here, \( x_i = 0 \) refers to a covered node and \( x_i = 1 \) to an uncovered one.

• The VC is minimum if the number of nodes with \( x_i = 1 \) is maximum.

• Therefore, take the limit \( \mu \to \infty \) (scale the field by \( h = \mu z \)).
Back to vertex covers (cont’d)

• The equation for the probability distribution becomes

\[ P_d(z) = \int_{-\infty}^{\infty} \prod_{l=1}^{d} (d z_l \sum_{d' = 0}^{\infty} p(d' | d) P_{d'}(z_l)) \delta(h - \mu - \sum_{l=1}^{d} \max(0, z_l)) \]

(11)

• By a clever Ansatz this equation can be solved
The solution

- Omitting details, details, and details, one arrives at

\[
\chi_c = 1 - \sum_{d=0}^{\infty} p_d (1 - \pi_{d-1})^{d-1} \left(1 + \frac{d - 2}{2} \pi_{d-1}\right) \tag{12}
\]

- The auxiliary variables \( \pi_d \) obey the self-consistency equation

\[
\pi_d = \sum_{d_l=0}^{\infty} p(d_l|d) (1 - \pi_{d_l}^{d_l}) \tag{13}
\]

- Physically \( \pi_d \) is the probability that an edge arriving at a vertex of degree \( d + 1 \) carries a constraint, i.e. is not covered.

- This is “the same” as the iteration for the fields \( h \) but now for vertex classes.
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**Numerics**

- **Protocol for solving** $\chi_c$
  - Iterate the equation for the $\pi_d$'s
  - Convergence $\implies$ evaluate $\chi_c$
  - Divergence $\implies$ RSB; no valid solution

- **Compare this to what the leaf-removal algorithm gives for test graphs**
  - Power-law degree distribution $p_d \propto d^{-\gamma}$ with $\gamma = 2.5$
  - Positive degree–degree correlations
    $$e_{dd'} = q_d [r \delta_{d,d'} + (1 - r) q_{d'}]$$
Generating graphs

- Randomize degrees $d_i$ independently for vertices using $p_d$’s
- Create a set $S$ of stubs such that each vertex $i$ appears in it $d_i$ times ($|S| = 2m$)
- For each edge, select first one endpoint randomly from $S$
- With probability $r$, select the other endpoint randomly from those with the same degree; otherwise randomly from all stubs
- Might lead to wanted correlations, but a bit suspicious (Catanzaro et al., PRE 71, 027103)
blue: $\gamma = 2.5$; red $\gamma = 3.0$; both $N = 10^6$
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- Real-world networks come typically with two kinds of correlations: disassortative \((r < 0)\) for technological nets and assortative \((r > 0)\) for social networks (Newman, PRE 67, 026126 (2003)).

- These nets also have quite often fat tails, thus power-law is good as a rough approximation (Dorogovtsev and Mendes, Advances in Physics 51, 1079 (2002)).

- So, solving for the VC of a real technological net should be easy with the leaf-removal algorithm.

- This is of importance since the deployment of a network traffic monitoring system that is capable of observing all edges is essentially a vertex cover.
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From GnuMap by Gregory Bray
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- There is an analytical replica-symmetric solution for the size of the minimum vertex cover in correlated random nets with arbitrary degree distribution.

- This comes in the form of a self-consistency equation; convergence of the iteration in different cases reveals if RS holds or not.

- Testing for prototype networks, positive correlations tend to break the replica symmetry.

- VC’s on real networks have applications in network traffic monitoring, for instance.
Thanks for attention