Phase transitions in combinatorial optimization problems
Course at Helsinki Technical University, Finland, autumn 2007
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4 Message passing algorithms

Basic idea: use statistical mechanic methods to find minimum VCs for give graphs $G = (V, E)$.
Bethe-Peierls approach: Solve models, which are defined on trees recursively.

Equivalent approaches in computer science/ information theory, but only for replica-symmetric (RS) cases (belief propagation (BP))

Here: also replica-symmetry broken (RSB) case ($\rightarrow$ survey propagation (SP))

Basic quantity ($i \in V$)

$$\pi_i \equiv \frac{|\{U \subset V \mid U \text{ is min. VC}, i \in U\}|}{|\{U \subset V \mid U \text{ is min. VC}\}|} \tag{1}$$

(how often i is covered for minimum VCs).

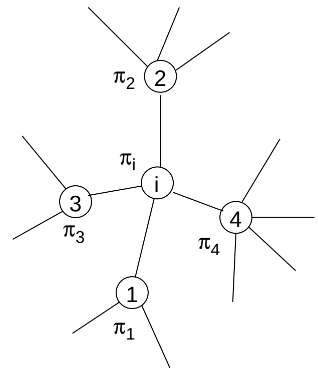
Construction of VCs, if π_is known (general outline):

- $\pi_i = 1$: covered backbone $\rightarrow$ cover!
- $\pi_i = 0$: uncovered backbone $\rightarrow$ uncover!
- $0 < \pi_i < 1$: Since vertices are not independent:
  Decimation Cover some vertices, remove them and adjacent edges
  Recalculate π for remaining graph
- Repeat until done

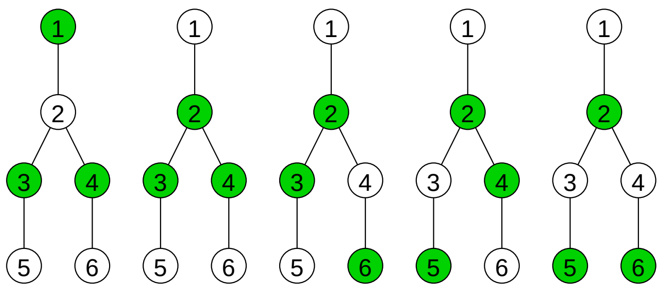
Main task: estimate π_is accurately without enumerating all VCs.

4.1 The cavity graph

First idea:
Give vertx i.
Assumption: for all $j \in N(i)$: π_j known
Calculate $\pi_i = 1 - \prod_{j \in N(i)} \pi_j$
$\rightarrow$ assumption: neighbors independent.
NOT true ! (if i is covered, <u>all</u> $j \in N(i)$: covered)



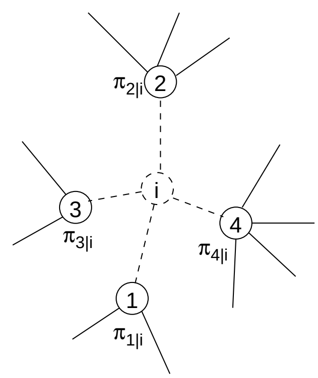
Example:



$\pi_1 = \frac{1}{5}, \pi_3 = \frac{3}{5}, \pi_4 = \frac{3}{5}$
but
$\pi_2 = \frac{4}{5} = \frac{100}{125} \neq \frac{116}{125} = 1 - \frac{9}{125} = 1 - \pi_1 \pi_3 \pi_4$

hence:

Definition: Cavity graph G_i: remove i and all edges $\{i, j\}$.
If graph <u>locally tree-like</u> (loops are infitely large)
$\rightarrow j_a, j_b \in N(i)$ are <u>almost</u> independent!
For finite graphs: algorithms = approximations



<u>Generalized probabilities</u>

$$\pi_{j|i} = \frac{|\{U \subset V_i \mid U \text{ is min. VC of } G_i,\ j \in U\}|}{|\{U \subset V_i \mid U \text{ is min. VC of } G_i\}|} \qquad (2)$$

4.2 Warning propagation

= Bethe-Peierls approach at $T = 0 \leftrightarrow$ min VCs.

Reducing π_i's:

$$\tilde{\pi}_i = \begin{cases} 0 & \text{if } \pi_i = 0 \\ * & \text{if } 0 < \pi_i < 1 \\ 1 & \text{if } \pi_i = 1 \end{cases} \qquad (3)$$

(similar $\tilde{\pi}_{j|i}$'s)
<u>joker state</u> $*$: "sometimes covered".

Aim: build self-consistent equations for $\{\tilde{\pi}_{j|i}\} \rightarrow$ define <u>messages</u>:
Intendet meaning:
$u_{j \to i}$ sent from $j \in V$ to $i \in N(j)$:

If j uncovered: $u_{j\to i} = 1$: "Attention, to cover our connecting edge you should be covered, or I have to change state" ("warning")
If, j: $u_{j\to i} = 0$: "You can be either covered or uncovered." ("trivial message")

Definition: For arbitray subset $U \subset V$

$$u_{j\to i}(U) = \begin{cases} 0 & \text{if } j \in U \\ 1 & \text{if } j \notin U \end{cases} \quad \text{for } \{i,j\} \in E \tag{4}$$

Extension to sets $\mathcal{M}$ of vertex subsets:

$$u_{j\to i}(\mathcal{M}) = \min_{U\in\mathcal{M}} u_{j\to i}(U) \ , \tag{5}$$

a warning is sent only if j is not contained in any $U \in \mathcal{M}$.
Special case: $\mathcal{M} = S_i$ = set of all min. VCs of G_i, then:

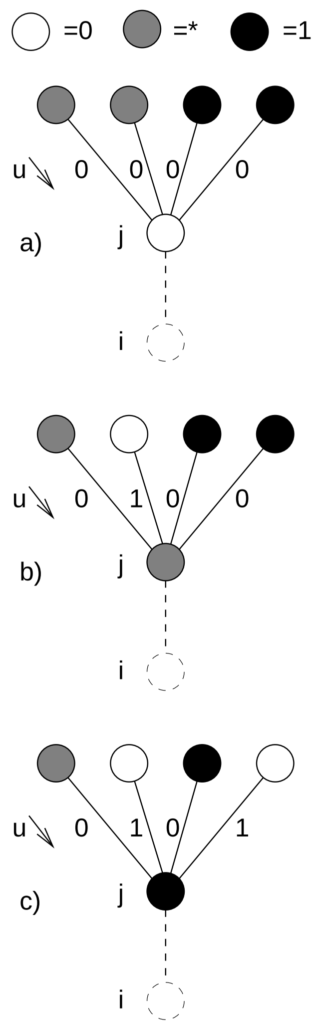
$$u_{j\to i}(\mathcal{S}_i) \equiv u_{j\to i}(\tilde{\pi}_{j|i}) = \begin{cases} 1 & \text{if } \tilde{\pi}_{j|i} = 0 \\ 0 & \text{if } \tilde{\pi}_{j|i} = * \\ 0 & \text{if } \tilde{\pi}_{j|i} = 1 \end{cases} \ . \tag{6}$$

On the other hand, $\tilde{\pi}_{j|i}$'s depend on messages:
Aim: constructing a min. VC by including j, given all neighbours k except i ($\to$ for G_i

a) for all $k \in N(j)$ there are min covers (of G_j !) where k is covered (i.e. messages 0)
$\to$ for a min VC (of G_i), j should not be covered

b) for all except $k \in N(j)$ except one, there are min covers (of G_j !) where k is covered (i.e. trivial messages 0), one k_0 is never covered (warning message 1)
$\to$ for a min VC of G_i, either j or k_0 covered

c) for at least two neigbours k_1, k_2 of j which are never covered (warning message 1)
$\to$ for a min VC of G_i, j must be covered



$$\Rightarrow \quad \tilde{\pi}_{j|i} = \begin{cases} 0 & \text{if } \sum_{k\in N(j)\setminus i} u_{k\to j} = 0 \\ * & \text{if } \sum_{k\in N(j)\setminus i} u_{k\to j} = 1 \\ 1 & \text{if } \sum_{k\in N(j)\setminus i} u_{k\to j} > 1 \end{cases} \ . \tag{7}$$

attention $u_{k\to j} = u_{k\to j}(\tilde{\pi}_{k|j})$!

Algorithm:
Initalize $2|E|$ warnings $u_{i\to j}$ randomly
Iterate (7) and (6) until convergence

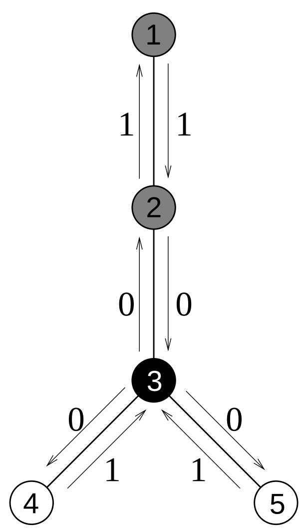
From converged cavity-graph warnings, non-cavity $\tilde{\pi}_j$ obtained in the same way, but for vertex j now all neighbours are considered:

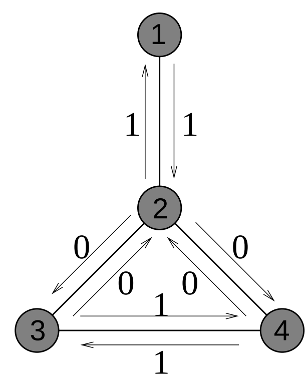
$$\tilde{\pi}_j = \begin{cases} 0 & \text{if } \sum_{k\in N(j)} u_{k\to j} = 0 \\ * & \text{if } \sum_{k\in N(j)} u_{k\to j} = 1 \\ 1 & \text{if } \sum_{k\in N(j)} u_{k\to j} > 1 \end{cases} . \tag{8}$$

Decimation: select one (or several) vertices l with highest$(0 < * < 1)$ $\tilde{\pi}_l$ and cover them. Remove l and adj. edges from graph. Restart iteration of (7) and (6).

Finally: VC comes out. Is minimum if $\tilde{\pi}_j$'s are correct (might fail in presence of loops).

Example: Success and failure of warning propagation





Left side, wanted $u_{4\to 3}$
4 has no incident edges in G_3 $\to$
$\tilde{\pi}_{4|3} = 0$ according Eq. (7)
$u_{4\to 3} = 1$ according Eq. (6)

Similarly $u_{5\to 3} = 1$, $u_{1\to 2} = 1$

For $2 \to 3$: One incoming message 1 (from 1)
$\to \tilde{\pi}_{2|3} = *$ Eq. (7)
$\to u_{2\to 3} = 1$ Eq. (6)

Edges leaving (3): at least one message 1 coming in
$\to \tilde{\pi}_{2|3} = *, 1$ Eq. (7)
$\to u_{3\to k} = 0\,\forall k$ Eq. (6)

For $2 \to 1$: One incoming message 0 (from 3)
$\to \tilde{\pi}_{2|1} = 0$ Eq. (7)
$\to u_{2\to 1} = 1$ Eq. (6)

Resulting $\tilde{\pi}_i$ are gray coded Eq. (8)
$\to$ vertex 3 will be covered (which is OK)

right side: a solution of the warning-propagation (check!!)
$\to$ there is no backbone
$\to$ vertex 1 could be decimated
$\to$ wrong !! (vertex 2 is ac bb $\to$ vertex 1 is auc bb) □

Note 1: Not only decimation might lead tonon-min. VC, also possible iteration does not converge (in RSB case, where many solutions exist $\to$ different vertices converge to different non-compatible values)
Note 2: Eqs. (6), (7), (8) can be used to calculate anaytical solutions for Erdős Réyni random graphs.

4.3 Extension

- Belief Propagation (BP):

  Same spirit as WP, but self-constent equations for $\pi_{j|i}, \pi_i \in [0, 1]$
  Decimation of vertices with largest π_i as above.

  See book.

- Survey Propagation (BP):

  Assumption: $\exists$ many solutions of self-consitent equations
  $\leftrightarrow$ many clusters of min VCs (RSB, $c > e$)
  $\to$ behavior might differ from cluster to cluster
  $\to$ introduce

  - $\hat{\pi}_i^{(1)}$: the fraction of clusters where vertex i takes state one
  - $\hat{\pi}_i^{(0)}$: the fraction of clusters where vertex i takes state zero
  - $\hat{\pi}_i^{(1)}$: the fraction of clusters where vertex i takes joker state $\star$.

  Analgous cavity quantites $\hat{\pi}_{j|i}^{(1)}$, $\hat{\pi}_{j|i}^{(0)}$ and $\hat{\pi}_{j|i}^{(*)}$.
  $\to$ Again solve self-consistent equations for cavity quanties, obtain non-cavity quantities +decimate

  See book.