2 Graphs

2.1 Basic Definitions

- (undirected) graph $G = (V, E)$: vertices $i \in V$ and undirected edges $\{i, j\} \in E \subseteq V^{(2)}$. Note $\{i, j\} = \{j, i\}$
- order $N = |V|$.
- size $M = |E|$.
- $i, j \in V$ are adjacent / neighboring if $\{i, j\} \in E$.
- $\{i, j\}$ is incident to $i$ and $j$.
- degree $\text{deg}(i)$ of $i$ = number of adjacent vertices. $i$ is isolated if $\text{d}(i) = 0$.
- path $E' = \{\{i_0, i_1\}, \{i_1, i_2\}, \ldots , \{i_{l-1}, i_l\}\} \subseteq E$, length $l = |E'|$. $E'$ goes from $i_0$ to $i_l$ and vice versa (end points).
- $i, j$ connected: $\exists$ path from $i$ to $j$.
- Connected component $V' \subseteq V$: all $i, j \in V'$ are connected.
- Matching $M \subseteq E$ such that no two edges in $M$ are incident to the same vertex.

Example: Graphs/ Matching

Graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 4\}, \{1, 5\}, \{2, 5\}, \{2, 6\}, \{3, 4\}\}$. 
Order $|V| = 6$, size $|E| = 5$.
Degrees, e.g. $d(1) = 2$, $d(3) = 1$. 
\[ E' = \{\{5, 1\}, \{1, 4\}, \{4, 3\}\} \text{: path from 5 to 3 of length 3.} \]

Left: matching \( M = \{\{1, 4\}, \{2, 5\}\} \).

Right: maximum-cardinality matching \( M = \{\{1, 5\}, \{2, 6\}, \{3, 4\}\} \).
\[ \square \]

- A graph \( G' = (V', E') \) is a subgraph of \( G \) if \( V' \subset V, \ E' \subset E \).
- complement graph \( G^C = (V, E^C) \): \( E^C = V^2 \setminus E = \{\{i, j\} \mid \{i, j\} \notin E\} \).

When edges have orientation:
- A directed graph \( G = (V, E) \): \( (i, j) \subset V \times V \): ordered pairs of vertices.
- directed path from \( i_0 \) to \( i_l \): \( E' = \{(i_0, i_1), (i_1, i_2), \ldots, (i_{l-1}, i_l)\} \subset E \)
- strongly connected component \( V' \): \( \forall i, j \in V' \), \( \exists \) a directed path from \( i \) to \( j \) and a directed path from \( j \) to \( i \).

### 2.2 Vertex-covers

- vertex cover (VC): Subset \( V_{vc} \subset V \) such that for each edge \( e = \{i, j\} \in E \)
  \( i \in V_{vc} \) or \( j \in V_{vc} \).
- \( V' \subset V \) arbitrary: elements \( i \in V' \) are called covered, also edges \( \{i, j\} \) with \( i \in V' \) or \( j \in V' \). Else uncovered.
- If all edges are covered, \( G \) also called covered.
- minimum vertex cover = vertex cover \( V_{vc} \) of minimum cardinality \( |V_{vc}| \).
- independent set of \( G \): \( I \subset V \) such that \( \forall i, j \in I \): \( \exists \) no edge \( \{i, j\} \in E \)
- clique of \( G \): \( Q \subset V \) such that \( \forall i, j \in Q \): \( \exists \{i, j\} \in E \).

Example: Vertex cover

Left: \( 1 \) and \( 2 \) covered (\( V' = \{1, 2\}\)), \( 3, 4, 5, 6 \) uncovered. \( \rightarrow \{1, 3\}, \{1, 4\}, \{2, 3\} \) covered, \( \{3, 4\}, \{3, 5\}, \{4, 6\}, \{5, 6\} \) uncovered. \( \rightarrow G \) not covered.

Right: \( 4 \) and \( 5 \) also covered. \( \rightarrow \{3, 4\}, \{3, 5\}, \{4, 6\}, \{5, 6\} \) now covered as well. \( \rightarrow G \) is covered by \( V_{vc} = \{1, 2, 4, 5\} \).
\( I = \{3, 6\} \) is an independent set.
\[ \square \]

Without proof: **Theorem:** For \( G = (V, E) \), \( V' \subset V \) the following three are equivalent.
(A) $V'$ is a vertex cover of $G$.

(B) $V \setminus V'$ is an independent set of $G$.

(C) $V \setminus V'$ is a clique of the complement graph $G^C$.

Def.:

- vertex-cover decision problem asks whether, there are VCs $V_{vc}$ of fixed given cardinality $X = |V_{vc}|$ $(x := X/N)$.

- cost function

$$H(V') = |\{(i, j) \in E \mid i, j \notin V'\}|,$$  \hspace{1cm} (1)

- constraint ground-state energy (optimization problem 1)

$$E(G, x) = Ne(G, x) = \min\{H(V') \mid V' \subset V, \ |V'| = xN\}$$  \hspace{1cm} (2)

- optimization problem 2: look for the minimum vertex cover, i.e. for a VC of minimum size

$$X_c(G) := N x_c(G) = \min\{|V'| \mid H(V') = 0\}.$$  \hspace{1cm} (3)

3 Algorithms for Vertex Cover

3.1 Heuristic algorithms

Find approximation of the true minimum VC.

1. Algorithm: Basic idea: cover as many edges as possible by using as few vertices as necessary.

algorithm greedy-cover($G = (V, E)$)

begin
	normalize $V_{vc} = \emptyset$;

while there are uncovered edges (i.e., $E \neq \emptyset$) do

begin

take one vertex $i$ of highest current degree $d_i$;

mark $i$ as covered: $V_{vc} = V_{vc} \cup \{i\}$;

remove from $E$ all edges $\{i, j\}$ incident to $i$;
Example:

Here it fails (shown is exact min. VC):

Empirically: cardinality differs usually only by a few percent from the exact minimum.

But: Greedy heuristic allows not for bound on the size of $V_{vc}$ compared to true minimum VC available.