Phase transitions in combinatorial optimization problems
Course at Helsinki Technical University, Finland, autumn 2007
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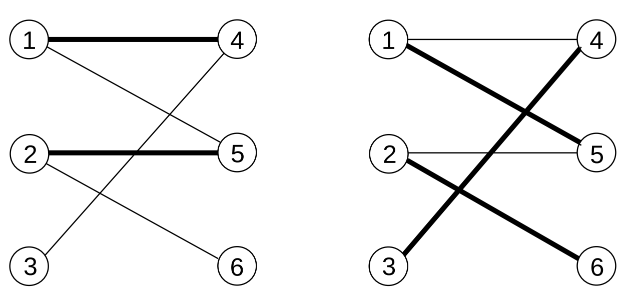
2 Graphs

2.1 Basic Definitions

- (undirected) graph $G = (V, E)$: vertices $i \in V$ and undirected edges $\{i, j\} \in E \subset V^{(2)}$. Note $\{i, j\} = \{j, i\}$
- order $N = |V|$.
- size $M = |E|$.
- $i, j \in V$ are adjacent / neighboring if $\{i, j\} \in E$.
- $\{i, j\}$ is incident to i and j.
- degree $\deg(i)$ of i = number of adjacent vertices. i is isolated if $d(i) = 0$.
- path $E' = \{\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{l-1}, i_l\}\} \subset E$, length $l = |E'|$. E' goes from i_0 to i_l and vice versa (end points).
- i, j connected: $\exists$ path from i to j.
- Connected component $V' \subset V$: all $i, j \in V'$ are connected.
- Matching $M \subset E$ such that no two edges in M are incident to the same vertex.

Example: Graphs/ Matching



Graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 4\}, \{1, 5\}, \{2, 5\}, \{2, 6\}, \{3, 4\}\}$.

Order $|V| = 6$, size $|E| = 5$.

Degrees, e.g. $d(1) = 2$, $d(3) = 1$.

$E' = \{\{5,1\}, \{1,4\}, \{4,3\}\}$: path from 5 to 3 of length 3.

Left: matching $M = \{\{1,4\},\{2,5\}\}$.

Right: maximum-cardinality matching $M = \{\{1,5\},\{2,6\},\{3,4\}\}$. □

- A graph $G' = (V', E')$ is a subgraph of G if $V' \subset V,\ E' \subset E$.
- complement graph $G^C = (V, E^C)$: $E^C = V^{(2)} \setminus E = \{\{i,j\} \mid \{i,j\} \notin E\}$.

When edges have orientation:

- A directed graph $G = (V, E)$: $(i,j) \subset V \times V$: ordered pairs of vertices.
- directed path from i_0 to i_l: $E' = \{(i_0, i_1),(i_1, i_2), \ldots,(i_{l-1}, i_l)\} \subset E$
- strongly connected component V': $\forall i, j \in V'$, $\exists$ a directed path from i to j and a directed path from j to i.

2.2 Vertex-covers

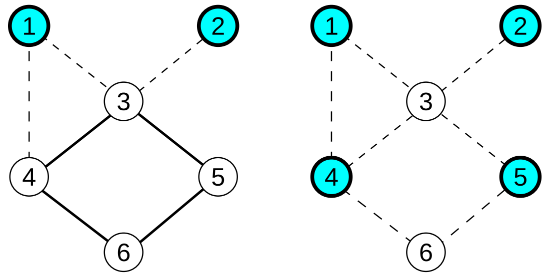
- vertex cover (VC): Subset $V_{vc} \subset V$ such that for each edge $e = \{i,j\} \in E$ $i \in V_{vc}$ or $j \in V_{vc}$.
- $V' \subset V$ arbitrary: elements $i \in V'$ are called covered, also edges $\{i,j\}$ with $i \in V'$ or $j \in V'$. Else uncovered.
- If all egdes are covered, G also called covered.
- minimum vertex cover = vertex cover V_{vc} of minimum cardinality $|V_{vc}|$.
- independent set of G: $I \subset V$ such that $\forall\, i,j \in I$: $\exists$ no edge $\{i,j\} \in E$
- clique of G: $Q \subset V$ such that $\forall\, i,j \in Q\ \exists\ \{i,j\} \in E$.

Example: Vertex cover

Left: 1 and 2 covered ($V' = \{1,2\}$), $3,4,5,6$ uncovered. $\rightarrow \{1,3\}$, $\{1,4\},\{2,3\}$ covered,$\{3,4\}$, $\{3,5\}$, $\{4,6\},\{5,6\}$ uncovered. $\rightarrow G$ not covered.

Right: 4 and 5 also covered. $\rightarrow \{3,4\}$, $\{3,5\}$, $\{4,6\},\{5,6\}$ now covered as well. $\rightarrow G$ is covered by $V_{\mathrm{vc}} = \{1,2,4,5\}$.
$I = \{3,6\}$ is an independent set. □

Without proof: **Theorem:** *For $G = (V, E)$, $V' \subset V$ the following three are equivalent.*



(A) V' is a vertex cover of G.

(B) $V \setminus V'$ is an independent set of G.

(C) $V \setminus V'$ is a clique of the complement graph G^C.

Def.:

- vertex-cover decision problem asks whether, there are VCs $V_{\rm vc}$ of fixed given cardinality $X = |V_{\rm vc}|$ $(x := X/N)$.
- cost function

$$H(V') = |\{\{i,j\} \in E \mid i,j \notin V'\}| \,, \tag{1}$$

- constraint ground-state energy (optimization problem 1)

$$E(G,x) = Ne(G,x) = \min\{H(V') \mid V' \subset V, \ |V'| = xN\} \tag{2}$$

- optimization problem 2: look for the minimum vertex cover, i. e. for a VC of minimum size

$$X_{\rm c}(G) := Nx_{\rm c}(G) = \min\{|V'| \mid H(V') = 0\} \,. \tag{3}$$

3 Algorithms for Vertex Cover

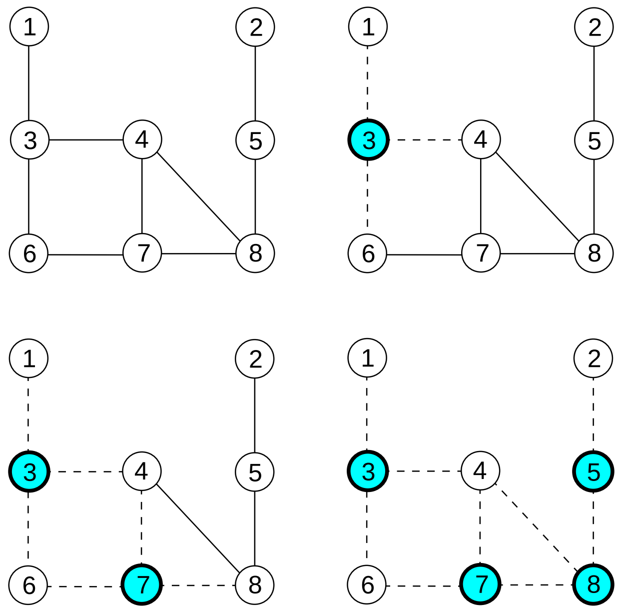
3.1 Heuristic algorithms

Find approximation of the true minimum VC.
1. Algorithm: Basic idea: cover as many edges as possible by using as few vertices as necessary.

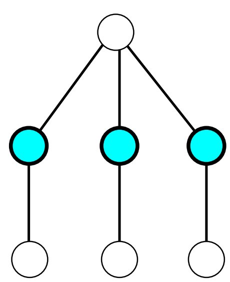
algorithm greedy-cover($G = (V,E)$
begin
  initialize $V_{\rm vc} = \emptyset$;
  while there are uncovered edges (i. e., $E \neq \emptyset$) **do**
  begin
    take one vertex i of highest current degree d_i;
    mark i as covered: $V_{\rm vc} = V_{\rm vc} \cup \{i\}$;
    remove from E all edges $\{i,j\}$ incident to i;

    end;
    return(V_{vc});
end

Example:



Here it fails (shown is exact min. VC):



Empirically: cardinality differs usually only by a few percent from the exact minimum.

But: Greedy heuristic allows <u>not</u> for bound on the size of V_{vc} compared to true minimum VC available.