Phase transitions in combinatorial optimization problems
Course at Helsinki Technical University, Finland, autumn 2007
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Lecture 1, 11. September 2007

1 Introduction

1.1 Technicalities

Survey: 5 CS students, 5 Physics students

Rules for credits:
Presentation (about 30-60 minutes using beamer) + written Summary (5-10 pages, best latex)
"Workshop" on Friday 12 October Test presentations on request (highly recommended, otherwise risk of no credit !!)
alternative: programming project (only 1 or 2, on request)
next lecture: Wed 12, 12-14
no lectures: 2nd week

tutorials: Fri 10-12 (from third week on)

- discussion/question hours
- special subjects on request (introduction to computational complexity, statistical physics, replica calculation, ...)
- presentations (12 October)

1.2 Examples

What is optimization:
In school: find maximum/minimum of function.

In Reality: many variables, often discrete: $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in X^n$, (e.g. $X = \{0,1\}, X = \mathbb{Z}$), $H(\underline{\sigma})$= cost function.
minimization problem

$$\textit{Find } \underline{\sigma} \in X^n\textit{, which minimizes } H!$$

$\max H = -\min(-H)$
constraints: decrease number of feasible solutions.

In economics: save resources/money/time etc.

Aim: Measure running time as a function of "input size".
Reminder: **Definition: O notation** Let T, g be functions from $\mathbb{N} \to \mathbb{R}$ We write $T(n) \in O(g(n)) :\Leftrightarrow \exists c > 0$ with $T(n) \leq cg(n)\ \forall$. We say: $T(n)$ is of order at most $g(n)$.

Worst-case running time: $T_{wc}(n) = \max_{\text{all instances } I \text{ withsize } n} T(I)$ (traditional CS)
Typical running time: Esemble of instances I of size n: average $T^*(n) := \langle T(I) \rangle_n$ or (better) median.

Example: Traveling Salesman Problem (TSP)

n cities distributed in a plane.
Find the shortest round-tour through all cities, each city only once.

$$X = \{1, 2, \dots, n\} \quad (1)$$

$$H(\underline{\sigma}) = \sum_{i=1}^{n} d(\sigma_i, \sigma_{i+1}) \quad (2)$$

$d(\sigma_\alpha, \sigma_\beta)$: distance between cities; $\sigma_{n+1} \equiv \sigma_1$. □

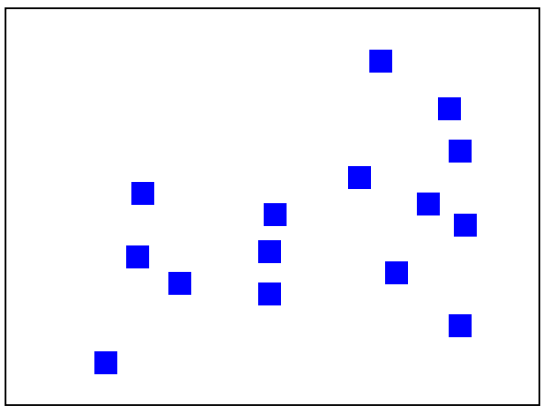

Figure 1: 15 cities in a plane.
Algorithms: worst-case running time increases exponentially with n

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In physics: ground state of magnetic systems, protein folding, data analysis, flux lines in superconductors

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Example: Ising Spin Glasses
Ising spin $\sigma_i = \pm 1$, simple lattice, nearest neighbor interactions (ferromagnetic/antiferromagnetic).

$$X = \{-1, 1\} \quad (3)$$

$$H(\underline{\sigma}) = -\sum_{<i,j>} J_{ij} \sigma_i \sigma_j \quad (4)$$

bonds $J_{ij} = \pm 1$. For each system: bonds fixed (quenched disorder)
→ Average over many samples.
Ground state=minimum of energy.

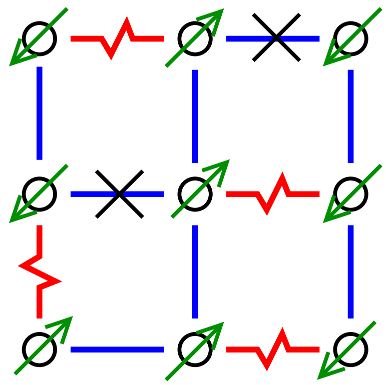


Figure 2: Two-dimensional spin glass. Solid lines = ferromagnetic interactions, jagged lines = antiferromagnetic interactions, small arrows = spins in ground state, crosses = unsatisfied interactions.

In 2d: fast algorithms, in 3d only slow algorithms. □

1.3 Why studying combinatorial optimizations using physics

Many practical optimization problems, also in physics
Optimum cost function ↔ minimum energy ↔ ground state ($T = 0$)
What makes a problem hard, related to glassiness?
Phase transitions ↔ easy/hard changes
Transformations to physical systems → methods from stat. mech.
Fast algorithms wanted: e.g. simulated annealing (old), survey propagation (new)
Analysis of algorithms.

Example: Phase transitions in the TSP

Plane area $A = L_x \times L_y$
Euclidean distances: $d(i,j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

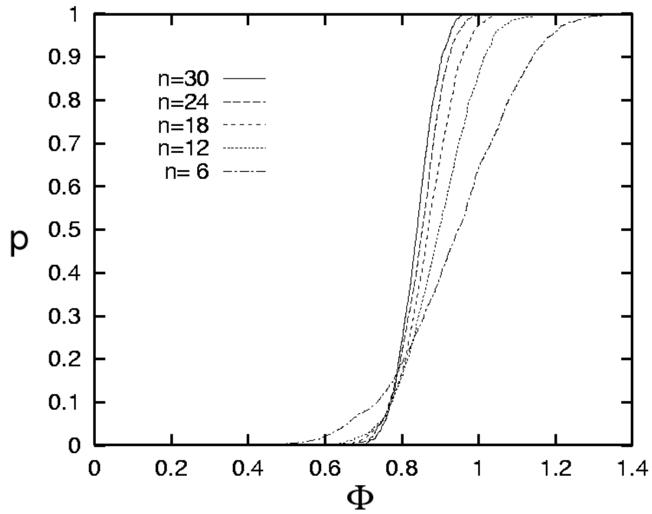
For each instance:

> *Is the shortest round trip through all cities shorter than a given length l?*

Solve exactly using branch-and-bound algorithm (see later)

Result: probability p of tour length $< l$ as function of rescaled length $\Phi = l/\sqrt{nA}$

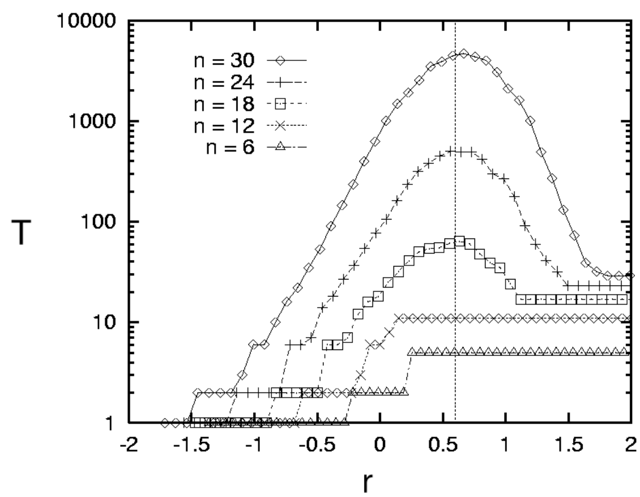
1. l scales with lengthscale of system, for constant n, i.e. $\sqrt{A}$. 2. l scales like A for constant density n/A. (Argument seems not work for $d > 2$, according analytic arguments.)



→ Drastical change at $\Phi = \Phi^\star \approx 0.78$ <u>phase transition</u>

rescaling: p as fct of $r = (\Phi - \Phi^\star)n^{2/3}$
→ data collaps (one curve)!

Running time: (mesured in terms of assigning city positions → machine independent)



Instances close to phase transition: hardest to solve!

Problem easy to solve for small/large values of l (i.e. Φ):

- Small l, even the two closest cities have a distance $> l$ → algorithm terminates
- Large l, even random permutation has total distance $< l$.

□

1.4 Bibliography

(Handout: B. Hayes, can't get no satisfaction)

- A.V. Aho, J.E. Hopcroft, J.D. Ullman: The design and analysis of computer algorithms, Addison-Wesley 1974 (ebay!)
- T.H. Cormen, S. Clifford, C. E. Leiserson, R. L. Rivest: Introduction to Algorithms, MIT Press 2001
- M.R. Garey und D. S. Johnson: Computers and intractability, Freeman, New York, 1979
- A.K. Hartmann und M. Weigt: Phase Transitions in optimization problems, Wiley-VCH, Berlin 2005
- B. Hayes: On the Threshold, American Scientist **91**, 12 (2003)
- K.H. Fischer and J.A. Hertz: Spin Glasses, Cambridge University Press, 1991
- S. Mertens: Computational Complexity for Physicists, Computing in Science & Engineering **4(3)**, 31 (2002)
- C.H. Papadimitriou und K. Steiglitz: Combinatorial Optimization, Prentice-Hall 1982

1.5 Subjects of Seminar

(details: see `http://www.tcs.hut.fi/Studies/T-79.7003/`)

- Phase transition in the Number-Partitioning Problem
- Modern Walk-SAT algorithms
- Calculation of phase boundary for vertex-cover problem (analytical) [presented by: Joni Parjinen]
- Calculation of typical running time of a branch-and-bound algorithm for the vertex-cover problem (analytical)
- Calculation of typical running times of WalkSAT algorithms (analytical)
- Generating hard but solvable SAT formulas (bit analytical)
- Phase transitions in generalized SAT problems
- Phase transition in minimizing spin-glass energies
- Vertex-cover on other graph ensmbles (bit analytical)
- Phase transition in ground states of random-field systems and running time of maximum-flow algorithms

- Combinatorial auctions [presented by: Olli Ahonen]
- Uniform sampling of local minima in Ising spin glasses [presented by: Petri Savola]