The Steiner Network Problem

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1. The Steiner Network Problem

- Also known as the *Survivable Network Design Problem*
- *Given:*
  - Undirected graph $G = (V, E)$ with nonnegative edge costs $c : E \to \mathbb{Q}^+$
  - Terminal connectivity requirement $r : \binom{V}{2} \to \mathbb{Z}^+$
- *Goal:*
  - Find minimum-cost subgraph of G that contains at least $r(u, v)$ edge-disjoint paths between each pair of terminals $\{u, v\}$.
- *Extension:*
  - Each edge $e \in E$ can have multiplicity $u_e \in \mathbb{Z}^+ \cup \{\infty\}$
  - General goal is to find a minimum-cost multigraph on V that satisfies the connectivity requirement. Each copy of edge e induces cost $c(e)$.

Linear programming formulation

- Each $S \subseteq V$ has associated cut requirement:

$$f(S) = \max\{r(u,v) \mid u \in S, v \in \overline{S}\}$$

- Recall notation for boundary of cut S:

$$\delta(S) = \{\{u,v\} \in E \mid u \in S, v \in \overline{S}\}$$

- Steiner network LP:

$$\begin{aligned}
\min \sum_{e \in E} c_e x_e & & \\
\text{s.t.} \sum_{e: e \in \delta(S)} x_e \geq f(S), & & S \subseteq V \\
x_e \in \{0, 1, \ldots, u_e\}, & & e \in E, u_e \neq \infty \\
x_e \in \mathbb{Z}^+, & & e \in E, u_e = \infty
\end{aligned}$$

LP relaxation

- Relaxed program:

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \sum_{e: e \in \delta(S)} x_e \geq f(S), \qquad S \subseteq V$$

$$u_e \geq x_e \geq 0, \qquad e \in E, u_e \neq \infty$$

$$x_e \geq 0, \qquad e \in E, u_e = \infty$$

- Note that program has exponentially many constraints, so cannot be solved in polynomial time in any obvious way.
- However, also a polynomial-sized LP can be developed.
- Alternately, a method such as the ellipsoid algorithm based on the notion of a polynomial-time separation oracle can be used. This is a subroutine that, given point x, either validates that x is a feasible solution or produces a violated constraint.

2. The Half-Integrality Property

- Our aim is to develop a 2-approximation algorithm for the Steiner Network Problem.
- This is easy for problems that have the half-integrality property: in any extremal feasible solution to the fractional LP, each variable x_e has value of the form $m \cdot (1/2)$, $m \geq 0$. (Thus in the case of binary variables, $x_e \in \{0, 1/2, 1\}$.)
- Extremal solution $\equiv$ not a convex combination of others.
- If the half-integrality property holds, one simply rounds up all the x_e in the fractional optimum. This at most doubles the total cost.

Half-integrality of Vertex Cover

- Consider e.g. the Vertex Cover Problem on a graph $G = (V, E)$, with vertex weights $c_v \in \mathbb{Q}^+$.
- Fractional LP formulation:

$$
\begin{aligned}
\min \sum_{v \in V} c_v x_v & \\
\text{s.t. } x_u + x_v \geq 1, & \qquad \{u, v\} \in E \\
x_v \geq 0, & \qquad v \in V
\end{aligned}
$$

Lemma 1. Let x be a feasible solution to the fractional Vertex Cover LP that is not half-integral. Then x is a convex combination of two (other) feasible solutions to the LP.

Proof. Partition the set of vertices that are not half-integral in x:

$$V_+ = \left\{ v \,\middle|\, \frac{1}{2} < x_v < 1 \right\}, \qquad V_- = \left\{ v \,\middle|\, 0 < x_v < \frac{1}{2} \right\}.$$

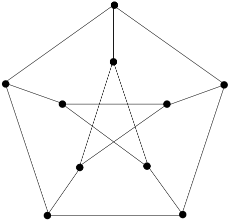
Then for any $\varepsilon > 0$, $x = \frac{1}{2}(y+z)$ for solutions y, z defined as follows:

$$y_v = \begin{cases} x_v + \varepsilon, & x_v \in V_+ \\ x_v - \varepsilon, & x_v \in V_- \\ x_v, & \text{otherwise} \end{cases} \qquad z_v = \begin{cases} x_v - \varepsilon, & x_v \in V_+ \\ x_v + \varepsilon, & x_v \in V_- \\ x_v, & \text{otherwise} \end{cases}$$

It can be verified that for small enough $\varepsilon > 0$, solutions $y, z \neq x$ are feasible for the fractional Vertex Cover LP. (The only nontrivial condition occurs when $x_u + x_v = 1$. But then the definition of V_+, V_- ensures that also $y_u + y_v = z_u + z_v = 1$ for any $\varepsilon > 0$.)

Half-integrality of Steiner Networks

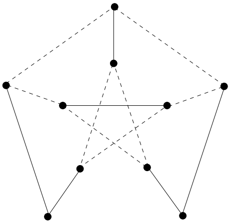
- Unfortunately, the Steiner Network Problem does not have the half-integrality property.
- Counterexample: the Petersen graph with connectivity requirement 1 between each pair of vertices.



- This has fractional optimal cost 5, achieved e.g. by solution with extent 1/3 of each edge. Any half-integral solution of cost 5 would have to pick two edges with extent 1/2 incident to each vertex, resulting in a Hamiltonian cycle of the graph. But the Petersen graph is nonhamiltonian.

An extremal solution on the Petersen graph

- Consider the following extremal optimum on the Petersen graph:



  Solid edges are picked to extent 1/2 and dotted edges to extent 1/4, for a total cost of 5.

- Note that although the solution is not half-integral, some edges are included to extent 1/2.
- This is in fact a general property of all extremal solutions to the Steiner Network Problem, and can be used to derive an iterated rounding algorithm for it, with approximation ratio 2.

3. Approximation by Iterated Rounding

- A requirement function f defined on the cuts of a graph $G = (V, E)$ is weakly supermodular if $f(V) = 0$ and for any two cuts $A, B \subseteq V$ at least one of the following holds:
  1. $f(A \cap B) + f(A \cup B) \geq f(A) + f(B)$
  2. $f(A \setminus B) + f(B \setminus A) \geq f(A) + f(B)$
- E.g. the original Steiner Network requirement function is weakly supermodular.
- **Theorem 2.** For any weakly supermodular requirement function f, any extremal feasible solution to the fractional Steiner Network LP satisfies $x_e \geq 1/2$ for at least one $e \in E$.

- This gives the first stage of the algorithm: find an extremal solution to the fractional SNLP, choose all edges e with $x_e \geq 1/2$, round their contributions up and remove them from the network. Then what?
- Given a set of (removed) edges H in a Steiner Network G, the residual requirement for a cut S is

$$f_H(S) = f(S) - |\delta_H(S)|,$$

  where $\delta_H(S)$ is the set of edges in H crossing S.
- **Lemma 3.** Let G be a Steiner network with requirement function f, and H a subgraph (set of edges) in G. If f is weakly supermodular, then so is f_H.

The iterated rounding algorithm

1. Set $H = \emptyset$, $f' = f$.
2. While $f' \neq 0$ do:
   - Find an extremal optimum x for the present fractional SNLP, with cut requirements f'.
   - For each edge e with $x_e \geq 1/2$, include $\lceil x_e \rceil$ copies of e in H, and decrement u_e by this number.
   - Update f': for $S \subseteq V$, $f'(S) = f(S) - |\delta_H(S)|$.
3. Output H.

Remaining questions

- Proof of Theorem 2?
- Proof of Lemma 3?
- Finding extremal optima to fractional SNLP?
- Approximation guarantee 2 also for iterated rounding?

Proof of Lemma 3

- A cut capacity function g on a graph $G = (V, E)$ is strongly submodular if $g(V) = 0$ and for any two cuts $A, B \subseteq V$ both of the following hold:
  1. $g(A \cap B) + g(A \cup B) \leq g(A) + g(B)$
  2. $g(A \setminus B) + g(B \setminus A) \leq g(A) + g(B)$
- **Lemma 3'.** For any graph H on vertex set V, the cut capacity function $|\delta_H(S)|$ is strongly submodular.
  Proof. By case analysis of Venn diagrams.
- **Lemma 3.** Let G be a Steiner network with requirement function f, and H a subgraph (set of edges) in G. If $f(S)$ is weakly supermodular, then so is $f_H(S) = f(S) - |\delta_H(S)|$.
  Proof. Straightforward from Lemma 3'.

Finding extremal optima to fractional SNLP

- By the ellipsoid method.
- Separation oracles are provided by max-flow techniques.

Approximation guarantee for iterated rounding

- By induction.