LP-techniques for facility location and $k$-medians
Chs. 24, 25

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Facility location problem
- A factor 3 approximation algorithm based on the primal-dual schema is presented.

$k$-Median problem
- A factor 6 approximation algorithm based on the previous algorithm is presented.
Facility location
Problem 24.1 (Metric uncapacitated facility location)

Let $G$ be a complete bipartite graph with bipartition $(F, C)$, where $F$ is a set of facilities and $C$ is the set of cities. Let $f_i$ be the cost of opening facility $i$, and $c_{ij}$ be the cost of connecting city $j$ to (opened) facility $i$. The connection costs satisfy the triangle inequality.

The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi: C \rightarrow I$ assigning cities to open facilities in such a way that the cost of opening facilities and connecting cities is minimized.

- The problem is related to, e.g., locating proxy servers on the internet, and clustering.
Let $y_i$ and $x_{ij}$ be indicator variables that denote whether facility $i$ is open and whether city $j$ is connected to facility $i$, respectively.

We get the following IP:

$$\begin{align*}
\text{minimize} & \quad \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\
\text{subject to} & \quad \sum_{i \in F} x_{ij} \geq 1, \quad j \in C \\
& \quad y_i - x_{ij} \geq 0, \quad i \in F, j \in C \\
& \quad x_{ij} \in \{0, 1\}, \quad i \in F, j \in C \\
& \quad y_i \in \{0, 1\}, \quad i \in F
\end{align*}$$
As usual, the LP-relaxation is obtained by letting the domain of variables $y_i$ and $x_{ij}$ be $[0, \infty[$:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\
\text{subject to} & \quad \sum_{i \in F} x_{ij} \geq 1, \quad j \in C \\
& \quad y_i - x_{ij} \geq 0, \quad i \in F, j \in C \\
& \quad x_{ij} \geq 0, \quad i \in F, j \in C \\
& \quad y_i \geq 0, \quad i \in F
\end{align*}
\]
The dual program uses variables $\alpha_j$ and $\beta_{ij}$:

maximize \[ \sum_{j \in C} \alpha_j \]

subject to \[ \alpha_j - \beta_{ij} \leq c_{ij}, \quad i \in F, j \in C \]
\[ \sum_{j \in C} \beta_{ij} \leq f_i, \quad i \in F \]
\[ \alpha_j \geq 0, \quad j \in C \]
\[ \beta_{ij} \geq 0, \quad i \in F, j \in C \]

The variable $\beta_{ij}$ can be viewed as the price paid by city $j$ towards opening facility $i$.

The variable $\alpha_j$ can be viewed as the total price paid by city $j$. 
In the primal-dual schema, relaxed versions of complementary slackness conditions are used to guide the algorithm.

The approximation factor is determined according to how much complementary slackness conditions have to relaxed for them to be satisfied by the solution obtained from the algorithm.

If a solution satisfies non-relaxed complementary slackness conditions, it is optimal.

Hence, complementary slackness conditions define desirable properties for the algorithm.
Complementary slackness conditions

Primal complementary slackness conditions

1. \( \forall i \in F, j \in C : x_{ij} > 0 \Rightarrow \alpha_j - \beta_{ij} = c_{ij} \)
   
   “The total price paid by the connected city goes towards making the connection and opening the facility.”

2. \( \forall i \in F : y_i > 0 \Rightarrow \sum_{j \in C} \beta_{ij} = f_i \)
   
   “Each open facility is fully paid for by the cities.”
Complementary slackness conditions

Dual complementary slackness conditions

1. \( \forall j \in C : \alpha_j > 0 \Rightarrow \sum_{i \in F} x_{ij} = 1 \)
   “All cities that pay anything must be connected to exactly one facility (with integral solutions).”

2. \( \forall i \in F, j \in C : \beta_{ij} > 0 \Rightarrow y_i = x_{ij} \)
   “A city does not contribute to opening any (open) facility besides the one that it is connected to.”
The algorithm is divided into two parts: Phase 1 and Phase 2.

Phase 1 finds a large dual feasible solution \((\vec{\alpha}, \vec{\beta})\) by changing only dual variables \(\alpha_j\) and \(\beta_{ij}\) such that feasibility is maintained at all times.

Phase 2 determines a primal (integral) feasible solution \((\vec{x}, \vec{y})\) based on the dual solution \((\vec{\alpha}, \vec{\beta})\).

The approximation factor is determined by observing how much the complementary slackness conditions have to be relaxed in order for them to be satisfied.
Primal-dual schema based algorithm — Phase 1

Algorithm 24.2 — Phase 1

- Set \((\vec{\alpha}, \vec{\beta}) = (\vec{0}, \vec{0})\), time to 0, and define all cities to be \textit{unconnected}.
- Do until all cities are \textit{connected}:
  - Simultaneously raise \(\alpha_j\) for each unconnected city \(j\) uniformly at unit rate, i.e., \(\alpha_j\) grows 1 in unit time.
  - If \(\alpha_j = c_{ij}\) for some edge \((i, j)\), declare this edge to be \textit{tight} and start also raising \(\beta_{ij}\) uniformly at unit rate until \(j\) gets connected.
  - If \(\sum_j \beta_{ij} = f_i\) for some facility \(i\), declare this facility \textit{temporarily open} and all unconnected cities having tight edges to \(i\) connected. Facility \(i\) is the \textit{connecting witness} of cities that are connected to it.
  - If an unconnected city \(j\) gets a \textit{tight} edge to a \textit{temporarily open} facility, declare \(j\) connected.
After Phase 1,
- \( \alpha_j - \beta_{ij} = c_{ij} \) for all tight edges \((i, j)\),
- \( \alpha_j < c_{ij} \) for all non-tight edges \((i, j)\),
- \( \sum_j \beta_{ij} = f_i \) for all temporarily open facilities \(i\),
- \( \sum_j \beta_{ij} < f_i \) for all non-temporarily open facilities \(i\).

Therefore, the fractional dual solution \((\vec{\alpha}, \vec{\beta})\) determined in Phase 1 is feasible.
The set $I$ of open facilities is picked from temporarily open facilities.

Let

- $F_t$ denote the set of open facilities,
- $T$ denote the subgraph of $G$ consisting of all “special” edges $(i, j)$ such that $\beta_{ij} > 0$,
- $T^2$ denote the graph that has edge $(u, v)$ iff there is a path of length at most 2 between $u$ and $v$ in $T$, and
- $H$ denote the subgraph of $T^2$ induced on $F_t$.

For city $j$, define $\mathcal{F}_j = \{ i \in F_t \mid (i, j) \text{ is special} \}$. 
Algorithm 24.2 — Phase 2

- Find any maximal independent set in $H$, say $I$.
- Iterate for all cities $j$:
  - If there is a facility $i \in F_j$ that is opened ($i \in I$):
    - Set $\phi(j) = i$ and declare city $j$ directly connected.
  - Else pick a tight edge $(i', j)$ such that $i'$ was the connecting witness for $j$.
    - If $i' \in I$, set $\phi(j) = i'$ and declare $j$ directly connected.
    - If $i' \notin I$, pick a neighbor $i$ of $i'$ such that $i \in I$. Set $\phi(j) = i$ and declare $j$ indirectly connected.
- Define a primal integral solution as follows:
  - Set $x_{ij} = 1$ iff $\phi(j) = i$.
  - Set $y_i = 1$ iff $i \in I$. 
After Phase 2,

- there is a facility $i$ such that $\phi(j) = i$ (i.e. $x_{ij} = 1$) for all cities $j$, 
- $\phi(j) = i$ (i.e. $x_{ij} = 1$) is set only whenever $i \in I$ (i.e. $y_i = 1$).

Therefore, the primal integral solution $(\vec{x}, \vec{y})$ determined in Phase 2 is feasible.
What about complementary slackness conditions?

### Dual complementary slackness conditions

1. \( \forall j \in C : \alpha_j > 0 \Rightarrow \sum_{i \in F} x_{ij} = 1 \)
2. \( \forall i \in F, j \in C : \beta_{ij} > 0 \Rightarrow y_i = x_{ij} \)

- Condition 1 is satisfied because \( x_{ij} = 1 \) is set for exactly one \( i \in F \) for all \( j \in C \).
- Condition 2 is satisfied because
  - \( \phi(j) = i \) if \( i \in F_j \) is open, and
  - \( \phi(j) \neq i \) if \( i \in F_j \) is not open.
What about complementary slackness conditions?

<table>
<thead>
<tr>
<th>Primal complementary slackness conditions</th>
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<tbody>
<tr>
<td>1. ( \forall i \in F, j \in C : x_{ij} &gt; 0 \Rightarrow \alpha_j - \beta_{ij} = c_{ij} )</td>
</tr>
<tr>
<td>2. ( \forall i \in F : y_i &gt; 0 \Rightarrow \sum_{j \in C} \beta_{ij} = f_i )</td>
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- Condition 2 is satisfied because only temporarily opened facilities are opened fully.
- Condition 1 is satisfied for directly connected cities because a directly connected city \( j \) is connected to its facility \( i \) through a tight edge \((i, j)\).
- Condition 1 is not necessarily satisfied for indirectly connected cities since an indirectly connected city might not be connected to its facility through a tight edge.
What about complementary slackness conditions?

- In order to satisfy all conditions, the first primal complementary condition must be relaxed for indirectly connected cities $j$ as follows:

\[(1/3)c_{\phi(j)}j \leq \alpha_j \leq c_{\phi(j)}j.\]

- This leads to an approximation algorithm that satisfies the inequality

\[\sum_{i \in F, j \in C} c_{ij}x_{ij} + 3 \sum_{i \in F} f_i y_i \leq 3 \sum_{j \in C} \alpha_j.\]

- Hence, the algorithm is a factor 3 approximation algorithm, but with a stronger inequality than typically.
Determination of the approximation factor

Denote by $\alpha_j^f$ and $\alpha_j^e$ the contributions of city $j$ to opening facilities and connection costs; $\alpha_j = \alpha_j^f + \alpha_j^e$.

If $j$ is indirectly connected, then $\alpha_j^f = 0$ and $\alpha_j^e = \alpha_j$.

If $j$ is directly connected, then $\alpha_j = c_{ij} + \beta_{ij}$, where $i = \phi(j)$.

Let $\alpha_j^f = \beta_{ij}$ and $\alpha_j^e = c_{ij}$. 
Lemma 24.4

Let $i \in I$. Then,

$$\sum_{j: \phi(j) = i} \alpha_j^f = f_i.$$ 

Proof.

Since $i$ is temporarily open at the end of Phase 1, it is completely paid for, i.e., $\sum_{j: \beta_{ij} > 0} \beta_{ij} = f_i$. If city $j$ has contributed to $f_i$, it must be directly connected to $i$. For each such city, $\alpha_j^f = \beta_{ij}$. Any other city $j'$ that is connected to facility $i$ must satisfy $\alpha_{j'}^f = 0$. The lemma follows.
Corollary 24.5

$$\sum_{i \in I} f_i = \sum_{j \in C} \alpha_j^f.$$ 

Lemma 24.6

*For an indirectly connected city $j$, $c_{ij} \leq 3\alpha_j^e$, where $i = \phi(j)$.***
Determination of the approximation factor

**Theorem 24.7**

The primal and dual solutions constructed by the algorithm satisfy

\[
\sum_{i \in F, j \in C} c_{ij} x_{ij} + 3 \sum_{i \in F} f_i y_i \leq 3 \sum_{j \in C} \alpha_j.
\]

**Proof.**

For a directly connected city \( j \), \( c_{ij} = \alpha^e_j \leq 3 \alpha^e_j \), where \( \phi(j) = i \).

Combining with Lemma 24.6, we get

\[
\sum_{i \in F, j \in C} c_{ij} x_{ij} \leq 3 \sum_{j \in C} \alpha^e_j.
\]

Adding to this the equality stated in Corollary 24.5 multiplied by 3 gives the theorem.
Denote $n_c = |C|$ and $n_f = |F|$.

Sort all the edges by increasing cost — this gives the order and the times at which edges go tight.

For each facility $i$, we maintain the number of cities that are currently contributing towards it, and the *anticipated time*, $t_i$, at which it would be completely paid for if no other event happens on the way.

$t_i$’s are maintained in a binary heap so we can update each one and find the current minimum in $O(\log n_f)$ time.
Running time

- During the execution of the algorithm, $t_i$’s in the binary heap are updated whenever a facility is completely paid for or an edge goes tight.
- Each edge $(i, j)$ will be considered at most twice: first, when it goes tight; second, when city $j$ is declared connected.

**Theorem 24.8**

Algorithm 24.2 achieves an approximation factor of 3 for the facility location problem and has a running time of $O(m \log m)$, where $m = n_c \times n_f$ is the number of edges.
$k$-Median
Problem 24.1 (Metric $k$-Median)

Let $G$ be a complete bipartite graph with bipartition $(F, C)$, where $F$ is a set of facilities and $C$ is the set of cities, and let $k$ be a positive integer specifying the number of facilities that are allowed to be opened. Let $c_{ij}$ be the cost of connecting city $j$ to facility $i$. The connection costs satisfy the triangle inequality.

The problem is to find a subset $I \subseteq F, |I| \leq k$ of facilities that should be opened and a function $\phi: C \rightarrow I$ assigning cities to open facilities in such a way that the total connecting cost is minimized.
Using indicator variables $y_i$ and $x_{ij}$, we get the following IP:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in F, j \in C} c_{ij} x_{ij} \\
\text{subject to} & \quad \sum_{i \in F} x_{ij} \geq 1, \quad j \in C \\
& \quad y_i - x_{ij} \geq 0, \quad i \in F, j \in C \\
& \quad \sum_{i \in F} -y_i \geq -k \\
& \quad x_{ij} \in \{0, 1\}, \quad i \in F, j \in C \\
& \quad y_i \in \{0, 1\}, \quad i \in F
\end{align*}
\]
LP-relaxation of the $k$-median problem

The LP-relaxation is obtained by letting the domain of variables $y_i$ and $x_{ij}$ be $[0, \infty[$:

- **minimize** \[ \sum_{i \in F, j \in C} c_{ij} x_{ij} \]
- subject to \[ \sum_{i \in F} x_{ij} \geq 1, \quad j \in C \]
- \[ y_i - x_{ij} \geq 0, \quad i \in F, j \in C \]
- \[ \sum_{i \in F} -y_i \geq -k \]
- \[ x_{ij} \geq 0, \quad i \in F, j \in C \]
- \[ y_i \geq 0, \quad i \in F \]
Introducing the variables $\alpha_j$ and $\beta_{ij}$, we obtain the dual program:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j \in C} \alpha_j - zk \\
\text{subject to} & \quad \alpha_j - \beta_{ij} \leq c_{ij}, \quad i \in F, j \in C \\
& \quad \sum_{j \in C} \beta_{ij} \leq f_i, \quad i \in F \\
& \quad \alpha_j \geq 0, \quad j \in C \\
& \quad \beta_{ij} \geq 0, \quad i \in F, j \in C \\
& \quad z \geq 0
\end{align*}
\]
The high-level idea

- Consider a facility location problem, where the opening cost for each facility is \( f_i = z \).
- By the strong duality theorem, the optimal fractional solutions \((\bar{x}, \bar{y})\) and \((\bar{\alpha}, \bar{\beta})\) satisfy
  \[
  \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} z y_i = \sum_{j \in C} \alpha_j.
  \]
- Suppose that the primal solution opens exactly \( k \) facilities, i.e., \( \sum_i y_i = k \).
The high-level idea

- We obtain the equality

\[ \sum_{i \in F, j \in C} c_{ij}x_{ij} = \sum_{j \in C} \alpha_j - zk. \]

- Hence, (\(\vec{x}, \vec{y}\)) and (\(\vec{\alpha}, \vec{\beta}, z\)) are optimal fractional solutions to the \(k\)-median problem.

- Now, suppose we use Algorithm 24.2 to find primal integral and dual feasible solutions (\(\vec{x}, \vec{y}\)) and (\(\vec{\alpha}, \vec{\beta}\)) to the facility location problem such that exactly \(k\) facilities are opened.
The high-level idea

- By Theorem 24.7, the solutions satisfy
  \[ \sum_{i \in F, j \in C} c_{ij}x_{ij} + 3zk \leq 3 \sum_{j \in C} \alpha_j. \]

- Hence, \((\vec{x}, \vec{y})\) and \((\vec{\alpha}, \vec{\beta}, z)\) are primal integral and dual feasible solutions that satisfy
  \[ \sum_{i \in F, j \in C} c_{ij}x_{ij} \leq 3 \left( \sum_{j \in C} \alpha_j - zk \right). \]

- Algorithm 24.2 is a factor 3 approximation algorithm for the \(k\)-median problem if the value of \(z\) can be chosen such that exactly \(k\) facilities are opened.
The high-level idea

- It is not known how to choose \( z \) such that exactly \( k \) facilities are opened.

- To overcome this problem, the algorithm is used to find solutions \((\vec{x}^s, \vec{y}^s)\) and \((\vec{x}^l, \vec{y}^l)\) to \( z_1 \) and \( z_2 \), respectively, such that \( k_1 < k, k_2 > k \), and \( z_1 - z_2 \leq c_{\text{min}}/(12n_c^2) \), where \( c_{\text{min}} \) is the length of the shortest edge.

- The values of \( z_1 \) and \( z_2 \) are determined by conducting a binary search on the interval \([0, nc_{\text{max}}]\), where \( n \) is the number of nodes and \( c_{\text{max}} \) is the length of the longest edge.
The high-level idea

- The feasible (fractional) solution

\[(\bar{x}, \bar{y}) = a(\bar{x}^s, \bar{y}^s) + b(\bar{x}^l, \bar{y}^l), \quad ak_1 + bk_2 = k,\]

opens exactly \(k\) facilities. Here,

\[a = (k_2 - k)/(k_2 - k_1),\]
\[b = (k - k_1)/(k_2 - k_1).\]

Lemma 25.2

*The cost of \((\bar{x}, \bar{y})\) is within a factor of \((3 + 1/n_c)\) of the cost of an optimal fractional solution to the \(k\)-median problem.*
An integral solution to the $k$-median problem is obtained from $(\vec{x}, \vec{y})$ using a randomized rounding procedure.

Let $A$ and $B$ be the sets of opened facilities in solutions $(\vec{x}^s, \vec{y}^s)$ and $(\vec{x}^l, \vec{y}^l)$, respectively.

For each facility in $A$, find the closest facility in $B$, and form a set $B' \subset B$ using this facilities; if $|B'| < |A|$, arbitrarily include additional facilities from $B - B'$ into $B'$ until $|B'| = |A| = k_1$.

Open the facilities in $A$ with probability $a = (k_2 - k)/(k_2 - k_1)$, and the facilities in $B'$ with probability $b = (k - k_1)/(k_2 - k_1)$.

Pick a set $D$ of cardinality $k - k_1$ from $B - B'$, and open the facilities in it.
The set of open facilities $I$ is either $A \cup D$ or $B' \cup D$.

Consider city $j$ that is connected to facilities $i_1 \in A$ and $i_2 \in B$.

If $i_1$ is open, set $\phi(j) = i_1$; if $i_2$ is open, set $\phi(j) = i_2$; otherwise, find the facility $i_3 \in B'$ that is closest to $i_1$ and set $\phi(j) = i_3$.

Denote by $\text{cost}(j)$ the connection cost for city $j$ in the fractional solution; $\text{cost}(j) = ac_{i_1j} + bc_{i_2j}$. 
Randomized rounding

**Lemma 25.3**

The expected connection cost for city $j$ in the integral solution, $E[c_{\phi(j)j}]$, is $\leq (1 + \max(a, b))\text{cost}(j)$. Moreover, $E[c_{\phi(j)j}]$ can be efficiently computed.

**Lemma 25.4**

Let $(\vec{x}^k, \vec{y}^k)$ denote the integral solution obtained to the $k$-median problem by this randomized rounding procedure. Then,

$$E\left[\sum_{i \in F, j \in C} c_{ij}x^k_{ij}\right] \leq (1 + \max(a, b))\left(\sum_{i \in F, j \in C} c_{ij}x_{ij}\right),$$

and, moreover, the expected cost of the solution can be found efficiently.
Randomized rounding

- Derandomization is done by opening those sets which minimize the previous expectation.
- The final approximation guarantee is 
  \[(1 + \max(a, b))(3 + \frac{1}{n_c}) \leq (2 + \frac{1}{n_c})(3 + \frac{1}{n_c}) < 6.\]
- The binary search will make \(O(L + \log n)\) probes, where 
  \(L = \log(\frac{c_{\text{max}}}{c_{\text{min}}}).\)

\[\text{Theorem 25.5}\]

*The algorithm achieves an approximation factor of 6 for the k-median problem, and has a running time of \(O((m \log m)(L + \log n))\).*
A Lagrangian relaxation technique for approximation algorithms

- A relaxation technique is a method in mathematical optimization for relaxing a strict requirement, e.g., by substituting it with another more easily handled requirement.
- Lagrangian relaxation technique consists of relaxing a (strict) constraint by moving it into the objective function, together with an associated Lagrangian multiplier $\lambda$.
- If the relaxed constrained is not satisfied, it induces a penalty on the objective function.
When applied to the $k$-median integer program, we obtain

$$\text{minimize } \sum_{i \in F, j \in C} c_{ij}x_{ij} + \lambda \left( \sum_{i \in F} y_i - k \right)$$

subject to

$$\sum_{i \in F} x_{ij} \geq 1, \quad j \in C$$

$$y_i - x_{ij} \geq 0, \quad i \in F, j \in C$$

$$x_{ij} \in \{0, 1\}, \quad i \in F, j \in C$$

$$y_i \in \{0, 1\}, \quad i \in F$$

This the facility location IP, where the cost of each facility has been set to $\lambda$, and an additional constant term $-\lambda k$ has been placed into the objective function.
Summary

- For the facility location problem, a factor 3 approximation algorithm based on the primal-dual schema was presented.
- This algorithm was used to construct a factor 6 approximation algorithm for the $k$-median problem.
- The primal-dual schema was used slightly differently here than in the previously presented algorithms.