Resolution Width and Interpolation

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Part I: From Short to Narrow Proofs

- Recall the rule of **resolution**: $$\frac{C \cup \{x\} \quad D \cup \{\overline{x}\}}{C \cup D}$$
- Resolution is a sound and complete refutation system for CNF formulas.
- The **length** of a refutation is number of clauses in it.
- The **width** of a refutation is the *maximal* number of literals in a clause.
- If a contradiction has a **short** refutation, then it has a **narrow** refutation.
- A refutation procedure can look for narrow proofs.

Resolution with Weakening

- The rule of **resolution**:
  - From $(C \vee x) \wedge (D \vee \overline{x})$ we can infer $C \vee D$.
  - $$\frac{C \cup \{x\} \qquad\qquad D \cup \{\overline{x}\}}{C \cup D}$$
- We will also allow **weakening** and **simplification**:
  - $$\frac{C}{C \cup D} \qquad\qquad \frac{}{\{1\}}$$
- Resolution with weakening and simplification is a sound and complete refutation system for CNF formulas.

Restriction of Clauses

- If F is a set or a sequence of clauses and x is a variable, then $F_{x=1}$ is the **restriction** of F by $x = 1$.

$$
\begin{array}{lcll}
F & & F_{x=1} & \\
\hline
C & \mapsto & C & \text{if } x, \overline{x} \notin C \\
C & \mapsto & \{1\} & \text{if } x \in C \\
C & \mapsto & C - \{\overline{x}\} & \text{if } \overline{x} \in C
\end{array}
$$

- We can derive $F_{x=1}$ from $F \cup \{x\}$ by resolution.
- $F_{x=0} := F_{\overline{x}=1}$.
- If Π is a resolution derivation from a clause set F, then $\Pi_{x=a}$ is a derivation from $F_{x=a}$.

Width of Resolution

- The width $w(C)$ of a clause C is the number of literals in it.
- The width $w(F)$ of a set or a sequence F of clauses is the **maximum** width of a clause in F.
- $w(F \vdash A)$ is the **minimum** width of a derivation of A from a clause set F.
- We are looking for a relationship between the width and length of a refutation.

Short Tree-Refutations Are Narrow

- Theorem 5.4.11. If there is a tree-like refutation of F consisting of at most 2^d lines, then $w(F \vdash \square) \leq w(F) + d$.
- Proof by induction on the number of variables n.
- Base case $n = 0$. The only possible refutation is $\langle \square \rangle$, which has length 1 and width 0.
- Induction step. Let F be an unsatisfiable set of clauses with $n > 0$ variables and let Π be a refutation of F of length at most 2^d.
- Lemma: If $w(F_{x=0} \vdash \square) \leq w$ then $w(F \vdash \{x\}) \leq w + 1$.

Proof Steps

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- 4. $F \vdash \{x\}$ in width $\leq w(F)+d$. Propagate
- 5. $F \cup \{x\} \vdash F_{x=1}$ in width $\leq w(F)$.

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- 4. $F \vdash \{x\}$ in width $\leq w(F)+d$. Propagate
- 5. $F \cup \{x\} \vdash F_{x=1}$ in width $\leq w(F)$.
- 6. $F \vdash \{\overline{x}\}$ in length $\leq 2^d$. Restrict

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- 4. $F \vdash \{x\}$ in width $\leq w(F) + d$. Propagate
- 5. $F \cup \{x\} \vdash F_{x=1}$ in width $\leq w(F)$.
- 6. $F \vdash \{\overline{x}\}$ in length $\leq 2^{d}$. Restrict
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- 4. $F \vdash \{x\}$ in width $\leq w(F)+d$. Propagate
- 5. $F \cup \{x\} \vdash F_{x=1}$ in width $\leq w(F)$.
- 6. $F \vdash \{\overline{x}\}$ in length $\leq 2^{d}$. Restrict
- 7. $F_{x=1} \vdash \Box$ in length $\leq 2^{d}$. Ind. hyp.
- 8. $F_{x=1} \vdash \Box$ in width $\leq w(F)+d$.

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- 4. $F \vdash \{x\}$ in width $\leq w(F) + d$. Propagate
- 5. $F \cup \{x\} \vdash F_{x=1}$ in width $\leq w(F)$.
- 6. $F \vdash \{\bar{x}\}$ in length $\leq 2^{d}$. Restrict
- 7. $F_{x=1} \vdash \Box$ in length $\leq 2^{d}$. Ind. hyp.
- 8. $F_{x=1} \vdash \Box$ in width $\leq w(F) + d$.
- 9. Combine 4, 5, and 8 to get a narrow refutation.

Results

- Let $L(F)$ (resp. $L_T(F)$) be the length of the shortest (tree-like) refutation of F.
- We just proved that $\log_2 L_T(F) \geq w(F \vdash \Box) - w(F)$.
- Dag-like proofs: $\log_2 L(F) = \Omega\left(\frac{(w(F \vdash \Box) - w(F))^2}{n}\right)$.
- $\Rightarrow$ Simple proofs for exponential lower bounds on resolution length.
- Ben-Sasson and Wigderson show a family of contradictions such that $w(F \vdash \Box) = O(1)$ and $L_T(F) = 2^{\Omega(|F| / \log |F|)}$.

A Refutation Procedure

- For a clause set F, repeat for width bound $w = 0, 1, \ldots$:
  - Add to F all resolvents of width $\leq w$.
  - Stop if $\Box \in F$.
- Running time $n^{O(w(F \vdash \Box))}$.
- For some formulas, exponentially faster than DPLL.

Part II: Interpolation

- What an interpolant is.
- How to obtain one.
- Tree-like resolution does not polynomially simulate resolution.

Interpolants

- Let $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ be disjoint vectors of propositional variables.
- Let $A(\mathbf{p}, \mathbf{q})$ and $B(\mathbf{p}, \mathbf{r})$ be propositional formulas such that $A \rightarrow \neg B$ is a tautology.
- Then there exists an **interpolant** $C(\mathbf{p})$ such that $A \rightarrow C$ and $C \rightarrow \neg B$ are tautologies.
- An interpolant always exists: let $C(\mathbf{a}) = \bigvee_{\mathbf{q}} A(\mathbf{a}, \mathbf{q})$.

Interpolants from Resolution

- Let $A(\mathbf{p},\mathbf{q})$ and $B(\mathbf{p},\mathbf{r})$ be CNF-formulas such that $A \wedge B$ is unsatisfiable.
- Theorem 5.4.13. If there is a resolution refutation for $A \wedge B$ of length k, then there exists a Boolean circuit $C(\mathbf{p})$ such that
  - $A \rightarrow C$ and $C \rightarrow \neg B$.
  - The circuit size of C is at most $kn^{O(1)}$.
  - If the variables in $\mathbf{p}$ occur only positively in A or only negatively in B, then C is a monotonic circuit.

Formulating *st*-Connectivity

- A graph cannot have a walk from s to t and at the same time a cut $[S,T]$ such that $s \in S$ and $t \in T$.
- Let $A(\mathbf{p},\mathbf{q}) =$ "$\mathbf{p}$ defines an undirected graph and $\mathbf{q}$ is a walk from 0 to $n+1$".
- Let $B(\mathbf{p},\mathbf{r}) =$ "$\mathbf{r}$ defines a cut between 0 and $n+1$".
- Let $p_{i,j} =$ "there is an edge from i to j", $q_{i,j} =$ "the ith node on the walk is j", and $r_i =$ "node i is in T".
- $B(\mathbf{p},\mathbf{r}) = \overline{r_0} \wedge r_{n+1} \wedge \bigwedge_{i \neq j} (\overline{r_i} \vee \overline{p_{i,j}} \vee r_j)$.
- The $p_{i,j}$ occur only negatively in B.

Proving *st*-Connectivity

- The total formula size is $O(n^3)$.
- There is a resolution refutation of size $O(n^4)$ of $A \wedge B$.
- Using the existence of a monotonic interpolant circuit, it can be shown that the size of a *tree-like* resolution refutation is $n^{\Omega(\log n)}$.
- The length of the shortest tree-like resolution can be superpolynomial in the length of a general resolution.
- Generally $L_T(F) = 2^{O\left(\frac{L(F)\log\log L(F)}{\log L(F)}\right)}$.

Summary

- The width of a refutation is the maximum number of literals in a clause.
- A tree-like refutation of length L_T implies a refutation of width $\log_2 L_T$.
- A general refutation of length L implies a refutation of width $O(\sqrt{n \log L})$.
- No narrow proof implies no short proof.
- Interpolants can be constructed from resolution proofs.
- Tree-like resolution does not polynomially simulate general resolution.