Resolutions Tseitin-Urquhart's Odd-Charged Graphs

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Contents

Introduction

- Tseitin associated propositional formulas with labeled undirected graphs

- He developed a technique for obtaining lower bounds for regular resolution refutations

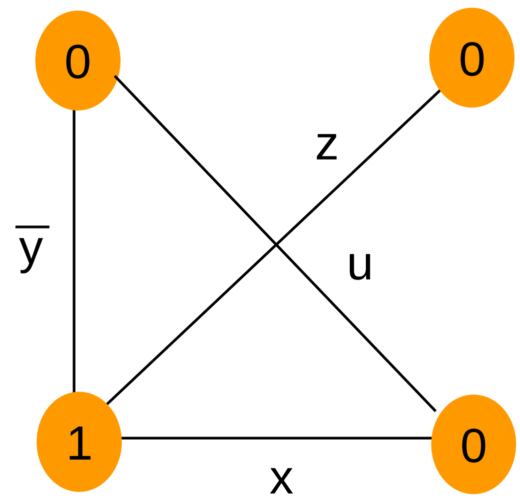
- Resolution allows, on any branch of the refutation tree, at most one resolution on any particular variable

Tseitin's odd-charged graphs (1/)

G = (V,E)   vertex, edges

Assign weight ***w(u)*={0,1}** to each node, **u**

Weight to each node is called a charge



Total charge ***w(G)*** is sum the sum modulo 2 of all the charges ***w(u)*** for

u є V

Edge are literals such that if edges e,é labeled with $\ell \ell^{l}$ respectively, then

$\{e, \neg e\}^{l} \cap \{\ell, \neg \ell^{l}\} = \phi$

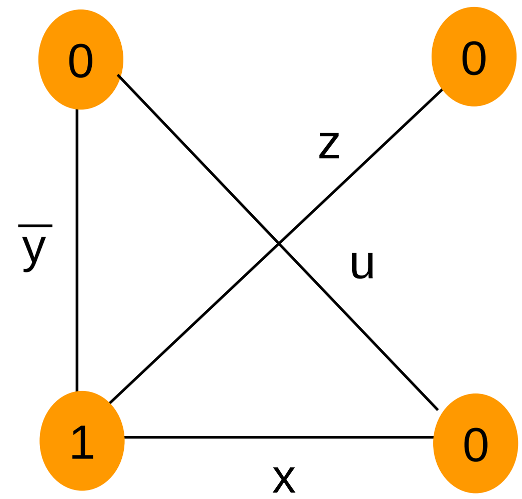
'i.e. if literal **α** labels edge **e,**

then neither **α** nor $\overline{\boldsymbol{\alpha}}$ can label another edge

Tseitin's odd-charged graphs (2/)

If $p1, \ldots, p_{deg(u)}$ are literals attached to a node *u*,
Let *E(u)* denote the equation

$p1 \oplus \ldots \oplus p_{deg(u)} = w(u)$, where *deg(u)* is that number of edges adjacent to *u*, *E(u)*



Let *C(u)* be a set of clauses formed by the conjunctive normal form of equation *E(u)*

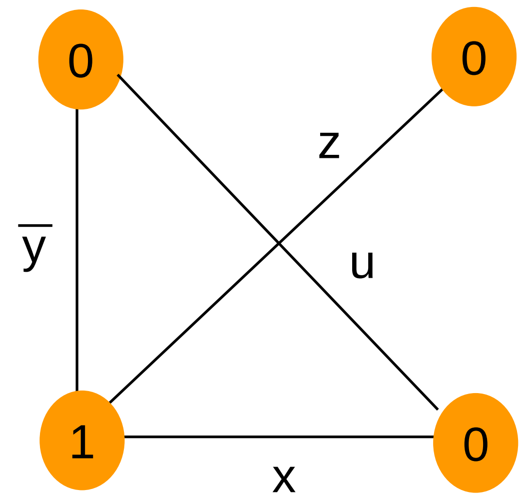
And let C(G) be the union over C ***є V*** of the sets *C(u) of claues*

It is clear that the number of sets of clause is $$\left|C(u)\right| = 2^{\deg(u)-1}$$

Tseitin's odd-charged graphs (3/)

Charge Equations are denoted as follows:

1. $\overline{y} \oplus u = 0$
2. $\overline{y} \oplus x \oplus z = 1$
3. $z = 0$
4. $x \oplus u = 0$



The Teistin clauses associated with the graph G are the clauses corresponding to the CNF formulation of the charged equations:

1. $\{\overline{u}, \overline{y},\}, \{u, y\}$
2. $\{x, y, \overline{z},\}, \{x, \overline{y}, z\}, \{\overline{x}, y, z\}, \{\overline{x}, \overline{y}, \overline{z},\}$
3. $\{\ \overline{z}\ \}$
4. $\{x, \overline{u}\},, \{\overline{x}, u\}$

$$|C(u)| = 2^{\deg(u)-1}$$

$$|C(u)| = 2^{3-1} = 2^2 = 4$$

e.g. Clause 2

When considering proof size, we are thus only interested in graph families of bounded degree

Tseitin's odd-charged graphs (4/)

We know there's a function which gives *0*

$$-f = (a \land b) \lor (\neg a \land \neg b)$$

Therefore $$f = \neg((a \land b) \lor (\neg a \land \neg b))$$

$$f = \neg(a \land b) \land \neg(\neg a \land \neg b)$$

$$f = (\neg a \lor \neg b) \land (a \lor b)$$

$$x \oplus y = 1 \equiv (\neg a \lor \neg b) \land (a \lor b)$$

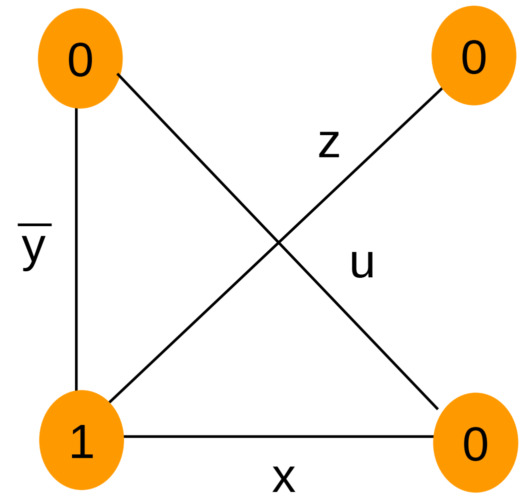
$$x \oplus y = 0 \equiv (a \land b) \lor (\neg a \land \neg b)$$

So, $$\bar{y} \oplus u = 0 \equiv (\neg(\neg y) \lor u) \land (\neg y \lor \neg u)$$

$$\bar{y} \oplus u = 0 \equiv (y \lor u) \land (\neg y \lor \neg u)$$

a	b	XOR
1	1	0
1	0	1
0	1	1
0	0	0

Tseitin’s odd-charged graphs (5/)



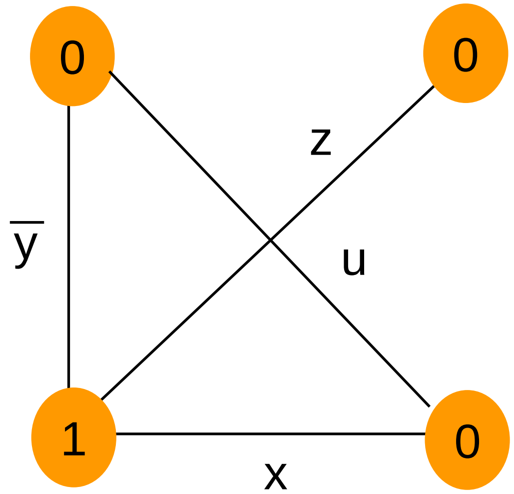
The key property of the odd-charged graphs is given by the the following:

The connected graph G is <u>odd-charged</u>

- If the sum mod 2 of all vertex charges is 1
- If and only if the clauses C(G) are unsatisfiable

C(G) is unsatisfiable <=> *w(G)* = 1

Tseitin's odd-charged graphs (6/)



For a *G* connected graph

C(G) is unsatisfiable <═> *w(G)* = 1

Proof:

Let *E(G)* denote the system *{E(u): u є V}*

***1st* we prove** *(<═),* Assume *w(G)= 1,*

The modulo 2 sum of the left hand is *0* since each literal is attached to two vertices

By assumption the right-hand side of *E(G) is 1.*

Hence there's no truth assignment satisfying *C(G)*

Tseitin's odd-charged graphs (7/)

Next we prove (⟹), Assume $w(G)= 0$

We show that $C(G)$ is satisfiable.

Let G_p be obtained from G by interchanging p and ¬p and by complementing the charges of the vertices incident to p.

Clearly, the system $E(G)$ and $E(G_p)$ have the same truth assignments

If nodes v, u have same charge=1,

Then $u=u_1,\ldots,u_r = v$ forming a path from u to v

For any truth assignment $б$, and any vertex u let $w_б(u)$ be the sum of modulo 2

Tseitin's odd-charged graphs (8/)

If N denotes the number of clauses of the formula Φn under consideration

E.g. number of clauses of PHP_n^{n+1} is $N_{\Phi_n} = \Theta\left(n^3\right)$

Then, Haken's lower bound shows that in fact the optimal resolution derivation of the empty clause from

$\neg PHP_n^{n+1}$ must have $2^{\Theta(N^{\frac{1}{3}})}$

Q. Are there examples of Φ_n with shortest resolution of size $2^{\Omega(n)}$

Where $|\Phi_n| = O(n)$?

Tseitin's odd-charged graphs (9/)

Φ_n we want to have a proof of size = 2^n

e.g. $N = |\Phi n| = 2n$, therefore $n = \dfrac{N}{2}$,

substituting n the proof size becomes $2^{N/2}$

But if $N = |\Phi n| = cn^3$, therefore $n = \sqrt[3]{\dfrac{N}{c}}$

Then proof size becomes $2^{\sqrt[3]{\frac{N}{c}}}$

As can be seen the proof size is not exponetial

Tseitin's odd-charged graphs (10/)

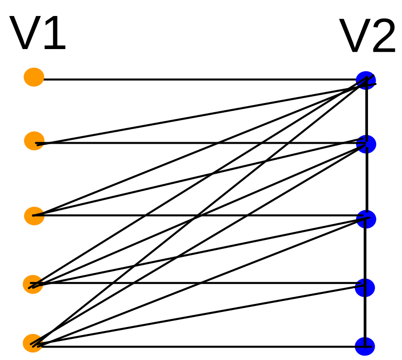
Let Hn be a bipartite graph consisting of two sides, each consisting of n=m nodes, such that each node is has a degree **≤** 5

There's a sequence Φ_n of valid formulas consisting *O(n)* many constant size clauses such that

Each $\neg\Phi_n$ has a polonomial-size $n^{O(1)}$ Frege refutation proof

But every refutation size has $2^{\Omega(n)}$

Gn



Each of the node of the new graph has a degree **≤ 7** and hence the clauses are of constant.

The formula ***Φn*** is the disjunction of the formulas ***C***, where ***C*** is a clause of ***C(Gn)***

Clearly is of size ***O(n)***

Tseitin's odd-charged graphs (11/)

There's a constant **d > 0** such that if *V1* is a set of nodes of size ≤ n/2 contained in one side of ***Gn*** and ***v2*** is the set of nodes in the opposite size of ***Gn*** connected to a node in ***V1*** by and edge, then

$$|V2| \geq (1+d)*|V1|$$

Note: **d ≤ 4**, Since Gn has a degree at most 5

1st show that ***Φn*** has a polynomial-size Frege proof. Denote the left(right) side of the charge equation E(u), Use propositional identities

$$p \oplus q \equiv \neg(p \leftrightarrow q)$$

$$(\neg p) \leftrightarrow (\neg q) \equiv p \leftrightarrow q$$

To convert $\bigoplus_{u \in V} \mathit{left}\ E(u) \leftrightarrow \bigoplus_{u \in V} \mathrm{right}\ E(u)$

To these into formulas consisting only of literals and the biconditional ↔ takes ***O(n)*** steps.

Using the associative and communicative laws of the biconditional we can move double literals to the front and eliminate these double occurrences.

Each of these steps takes $O(n^2)$ steps thus yielding the desired contradiction

0 ↔ **1** in a total of $O(n^3)$ steps, each of length ***O(n)***

Hence the size of Frege proof is $O(n^4)$

Tseitin's odd-charged graphs (12/)

Next we prove the lower bound of resolution refutations of $C(G_n)$

I'll leave this proof for homework

After proving this you can show that

For any partial truth assignment $\boldsymbol{\sigma}$ the clause $\boldsymbol{C}$ contains $[\boldsymbol{dn/16}]$ literals

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Schöning Expander Graphs and Resolutions (1/)

Shöning's simplification of Urqurt exponential lower bound of Tsein's refutations formulas for certain class of odd-charged graphs

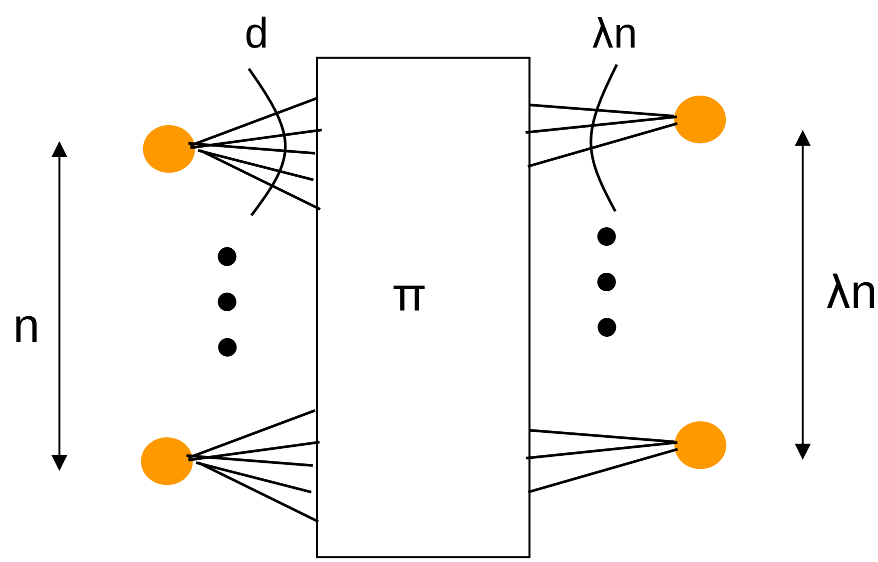
Shöning proof uses two basic ideas

1. By setting chosen literals to 0 and 1 appropriately, kill off all large clauses
2. By appropriately toggling certain critical truth assignments, prove there's a remaining large clause (having many literals)

The ability to so toggle certain truth assignments uses the existence of certain expander graphs whose existence is proved by a new probabilistic construction.

Schöning Expander Graphs and Resolutions (1/)

- *A* **(n, d, λn)**-graph *is a bipartite (multi-) graph with* ***n*** *vertices on the left side and* ***λn*** *vertices on the right side.*
- *Each vertex on the left has degree* ***d****, and each vertex on the right has degree* ***d/ λ***
- *A* ***(n, d, λn)*** *-graph is (****α, β****)*-expanding *if every subset* S *of vertices on the left side of size* ***n*** *has more than* ***λn*** *neighbors on the right side,* ***0 < α ≤ β < 1*** *(Notice that β might depend on α)*



There exits a family of undirected degree graphs Gn= (Vn, En) where

Vn = {1,…,n} such that every refutation of the related CNF Tseitin formula ***Φn*** has at least $2^{\beta n}$ clauses

Summary

- Tseitin-Urquhart's graphs, charge equations and clauses associated with the graph
- Key property of the odd-charged graphs
- Proof of the properties and by obtaining lower bounds for regular resolution refutations
- Shöning's basic ideas for simplification of Urqurt exponential lower bound of Tsein's refutations formulas for certain class of odd-charged graphs