ISOPERIMETRIC PROBLEMS

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THE ISOPERIMETRIC PROBLEM

Among all closed curves of length \( \ell \), which one encloses the maximum area?

For graphs: separator problems (vertex and edge cuts) — relations between the cut sizes and the sizes of the separated parts
Volume and Boundary

- **Notation**: graph $G = (V, E(G))$, set $S \subset V$, $|V| = n$
- **Volume**: $\text{vol } S = \sum_{v \in S} d_v$
- **Edge boundary**: $\partial S = \{\{u, v\} \in E(G) \mid u \in S, v \notin S\}$
- **Vertex boundary**: $\delta S = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$
**RELATED PROBLEMS**

Given a fixed integer $m$, find a subset $S$ with $m \leq \text{vol } S \leq \text{vol } \bar{S}$ s.t.

1. the boundary $\partial S = \{\{u, v\} \in E(G) \mid u \in S, v \notin S\}$ contains as few edges as possible

2. the boundary $\delta S = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$ contains as few vertices as possible
CHEEGER CONSTANT

\[ h_G = \min_S \frac{|\partial S|}{\min \{\text{vol } S, \text{vol } \bar{S}\}} \]

From the definition, we get for \( S \) s.t. \( \text{vol } S < \text{vol } \bar{S} \) that \( |\partial S| \geq h_G \cdot \text{vol } S \).

Also, \( G \) is connected iff \( h_G > 0 \).
**Vertex expansion**

\[ g_G = \min_S \frac{|\delta S|}{\min\{\text{vol} S, \text{vol} \bar{S}\}} \]

**Regular graphs:** \[ g_G(S) = \frac{|\delta S|}{\min\{|S|, |\bar{S}|\}} \]

**Definition:** (volume replaced by unit measure)

\[ \bar{g}_G = \min_S \frac{|\delta S|}{\min\{|S|, |\bar{S}|\}} \]
**Lemma:** $2h_G \geq \lambda_1$

Setup for the proof:

- $C$ is a cut that achieves $h_G$
- $C$ splits $V$ into sets $A$ and $B$
- **Definition:** $f(v) = \begin{cases} 
  \frac{1}{\text{vol} A}, & \text{if } v \in A, \\
  -\frac{1}{\text{vol} B}, & \text{if } v \in B 
\end{cases}$
Expression for $\lambda_1$

$$\lambda_1 = \lambda_G = \inf_{f \perp T_1} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v (f(v))^2 d_v}$$
**Proof of** \(2h_G \geq \lambda_1, \text{ Part } \frac{1}{2}\)

Using the definition of \(\lambda_1\) with definitions of \(\text{vol } S, C\) and \(f\), we get the result. First we simply “partition” the expression using \(A\) and \(B\):

\[
\lambda_1 = \inf_{f \perp T_1} \frac{\sum_{v} (f(v))^2 d_v}{\sum_{u \sim v} (f(u) - f(v))^2}
\]

\[
= \frac{\sum_{u \in A, v \in B} (f(u) - f(v))^2 + \sum_{u \in A, v \sim A} (f(u) - f(v))^2 + \sum_{u \in B, v \in B} (f(u) - f(v))^2}{\sum_{v \in A} (f(v))^2 d_v + \sum_{v \in B} (f(v))^2 d_v}
\]
Proof continues, part $\frac{2}{2}$

We use the definitions of $f$ and vol:

$$
\lambda_1 = \frac{\sum_{u \in A, v \in B} \left( \frac{1}{\text{vol} A} + \frac{1}{\text{vol} B} \right)^2 + 0 + 0}{\sum_{v \in A} \frac{d_v}{(\text{vol} A)^2} - \sum_{v \in B} \frac{d_v}{(\text{vol} B)^2}}
$$

$$
= \left| C \right| \left( \frac{1}{\text{vol} A} + \frac{1}{\text{vol} B} \right)^2
$$

$$
= \left| C \right| \left( \frac{1}{\text{vol} A} + \frac{1}{\text{vol} B} \right)
$$
**Theorem:** $\lambda_1 > \frac{h_G^2}{2}$

Setup for proof:

- **vertex labels** $v_1, v_2, \ldots, v_n$ such that $f(v_i) \leq f(v_{i+1})$  
  $(1 \leq i \leq n - 1)$

- **w.l.o.g.**  
  \[
  \sum_{f(v)<0} d_v \geq \sum_{f(v)\geq0} d_v
  \]

- **cuts** $C_i = \{v_j, v_k\} \in E(G) \mid 2 \leq j \leq i < k \leq n$, $1 \leq i \leq n$
DEFINITIONS FOR THE PROOF

• **Definition:** $\alpha = \min\limits_{1 \leq i \leq n} \frac{|C_i|}{\min\left\{\sum\limits_{j \leq i} d_j, \sum\limits_{j > i} d_j\right\}}$

• By definition $\alpha \geq h_G$ (divisors are the volumes of the parts)

• $V_+ = \{v \in V \mid f(v) \geq 0\}$

• $E_+ = \{\{u, v\} \in E(G) \mid u \in V_+, v \in V\}$

• $g(v) = \begin{cases} f(v), & \text{if } v \in V_+, \\ 0, & \text{otherwise} \end{cases}$
HARMONIC EIGFN $f$ OF $\mathcal{L}$ WITH EIGVAL $\lambda_1$

For any $v \in V$, it holds for $f$ that

$$\frac{1}{d_v} \sum_{u \sim v} (f(v) - f(u)) = \lambda_1 f(v)$$

$$\Rightarrow \lambda_1 = \frac{1}{d_v f(v)} \sum_{u \sim v} (f(v) - f(u)) \quad (\dagger)$$

(a lemma from the previous chapter)
Proof of the theorem, part $\frac{1}{8}$

Substituting $\lambda_1 = (\dagger)$ and summing over $V_+$

$$
\lambda_1 = \frac{1}{\int_{V_+} f(v) dv} \sum_{v \sim v} (f(v) - f(u))
$$

$$
= \frac{\sum_{v \in V_+} f(v) \sum_{\{u, v\} \in E_+} (f(v) - f(u))}{\int_{V_+} (f(v))^2 dv}
$$

because for any subset $S \subseteq V$, we have

$$
\lambda_1 f(v) dv = \sum_{u \sim v} (f(v) - f(u))
$$

$$
\lambda_1 (f(v))^2 dv = f(v) \sum_{u \sim v} (f(v) - f(u))
$$

$$
\lambda_1 \sum_{v \in S} (f(v))^2 dv = \sum_{v \in S} f(v) \sum_{u \sim v} (f(v) - f(u))
$$
PROOF OF THE THEOREM, PART $\frac{2}{8}$

From the def's of $g, V_+$ and $E_+$ (as $(f(u))^2 > 0$ and $g(v) \geq f(v)$),

$$
\lambda_1 = \frac{\sum_{v \in V_+} f(v) \sum_{\{u,v\} \in E_+} (f(v) - f(u))}{\sum_{v \in V_+} (f(v))^2 d_v}
$$

$$
= \frac{\sum_{v \in V_+} ((f(v))^2 - f(v)f(u))}{\sum_{v \in V_+} (f(v))^2 d_v}
$$

$$
> \frac{\sum_{\{u,v\} \in E_+} (g(u) - g(v))^2}{\sum_{v \in V} (g(v))^2 d_v}
$$

(△)
Using the Cauchy-Schwarz inequality \((\sum x_i y_i)^2 \leq (\sum x_i^2)(\sum y_i^2)\) with \(x_i = |g(u) - g(v)|\) and \(y_i = g(u) + g(v)\), we get

\[
\lambda_1 > \frac{\sum_{\{u,v\} \in E_+} (g(u) + g(v))^2}{\sum_{\{u,v\} \in E_+} (g(u) + g(v))^2} \cdot \frac{\sum_{\{u,v\} \in E_+} (g(u) - g(v))^2}{\sum_{v \in V} (g(v))^2 d_v} \tag{\ast}
\]

\[
\geq \sum_{u \sim v} \frac{|(g(u) - g(v))| (g(u) + g(v))}{2 \left( \sum_v (g(v))^2 d_v \right)^2}
\]
Proof of the theorem, part \( \frac{4}{8} \)

Now using \((a + b)(a - b) = a^2 - b^2\), we get

\[
\lambda_1 \geq \sum_{u \sim v} \left| (g(u) - g(v)) \right| \left( g(u) + g(v) \right) \]

\[
\geq \frac{\sum_{u \sim v} \left| (g(u) - g(v)) \right| \left( g(u) + g(v) \right)}{2 \left( \sum_{v} (g(v))^2 d_v \right)^2}
\]

\[
\geq \frac{\left( \left| (g(u))^2 - (g(v))^2 \right| \right)^2}{2 \left( \sum_{v} (g(v))^2 d_v \right)^2}
\]
Now from the definition of $C_i$ and "partitioning" the edges to "steps" over the cuts $C_i$, we continue

$$\lambda_1 \geq \frac{2 \left( \sum_{u \sim v} \left| (g(u))^2 - (g(v))^2 \right| \right)^2}{2 \left( \sum_v (g(v))^2 d_v \right)^2} \cdot \sum_i \left| (g(v_i))^2 - (g(v_{i+1}))^2 \right| \cdot |C_i|^2$$
Proof of the theorem, part $\frac{6}{8}$

Using the definition of $\alpha$ together with the fact that
\[ \sum_{f(v)<0} d_v \geq \sum_{f(v)\geq 0} d_v \]
and the vertex ordering, we get
\[
\lambda_1 \geq \frac{\left( \sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \cdot |C_i| \right)^2}{2 \left( \sum_v (g(v))^2 d_v \right)^2} \cdot \frac{\left( \sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \cdot \alpha \sum_{j>i} d_j \right)^2}{2 \left( \sum_v (g(v))^2 d_v \right)^2}.
\]
PROOF OF THE THEOREM, PART $\frac{7}{8}$

\[
\left( \frac{\sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \sum_{j>i} d_j}{\left( \sum_v (g(v))^2 d_v \right)^2} \right)^2 = \frac{\sum_{i=0}^{n-1} (g(v_{i+1}))^2 d_{i+1}}{\sum_{v=1}^{n} (g(v))^2 d_v} = 1
\]

as when we multiply the nominator “open”, all but one of the \((g(v_{i+1}))^2\) cancel out, appearing both positive and negative, except for once for \(j = i + 1\), which leaves the same summation than we have in the denominator.
Now we simply take out \( \alpha^2 \) and use the previous observation and the definition of \( \alpha \) to complete the proof:

\[
\lambda_1 \geq \frac{\left( \sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \cdot \alpha \sum_{j>i} d_j \right)^2}{2 \left( \sum_v (g(v))^2 d_v \right)^2} = \frac{\alpha^2}{2} \geq \frac{h_G^2}{2}.
\]
Putting together the lemma and the theorem, we have

\[ 2h_G \geq \lambda_1 > \frac{h_G^2}{2}. \]
**IMPROVEMENT:** \( \lambda_1 > 1 - \sqrt{1 - h_G^2} \)

From the proof of the previous theorem we have \( \lambda_1 = (\triangle) \) and we define \( W = (*) \):

\[
\lambda_1 = \frac{\sum_{v \in V_+} f(v) \sum_{\{u,v\} \in E_+} (f(v) - f(u))}{\sum_{v \in V_+} (f(v))^2 d_v} > \frac{\sum_{\{u,v\} \in E_+} (g(u) - g(v))^2}{\sum_{v \in V} (g(v))^2 d_v} = W
\]
PROOF OF THE SECOND THEOREM

Again we extend and use some already familiar tricks (plugging in the def. of $W$ itself):

$$W = \frac{\sum_{\{u,v\} \in E_+} (g(u) + g(v))^2}{\sum_{\{u,v\} \in E_+} (g(u) + g(v))^2} \cdot \frac{\sum_{\{u,v\} \in E_+} (g(u) - g(v))^2}{\sum_{v \in V} (g(v))^2 d_v} \cdot \left( \sum_{u \sim v} |(g(u))^2 - (g(v))^2| \right)^2 \geq \frac{\left( \sum_{v} (g(v))^2 d_v \right)^2}{\left( \sum_{v} (g(v))^2 d_v \right) \cdot \left( 2 \sum_{v} (g(v))^2 d_v - W \sum_{v} (g(v))^2 d_v \right)}$$
Proving the nominator just as in the previous proof, simple factorization of the denominator gives

\[ W \geq \left( \sum_i |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot |C_i| \right)^2 \]

\[ \geq \left( \sum_i |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot \alpha \sum_{j>i} d_j \right)^2 \]

\[ = \frac{\alpha^2}{2 - W} \]
**Intermediate Result:** \[ W \geq \frac{\alpha^2}{2 - W} \]

\[ \Rightarrow W^2 - 2W + \alpha^2 \leq 0. \]

Solving the zeroes gives \( W \geq 1 - \sqrt{1 - \alpha^2} \).

By definitions of \( W \) and \( \alpha \), we have \( \lambda_1 > W \) and \( \alpha \geq h_G \). Hence **we have proved the theorem** \( \lambda_1 > 1 - \sqrt{1 - h_G^2} \). Note that

\[ \frac{h_G^2}{2} < 1 - \sqrt{1 - h_G^2} \]

whenever \( h_G > 0 \) (i.e., for any connected graph), meaning that this is **always an improvement** to the previous lower bound.
CONSTRUCTIONAL “COROLLARY”

In a graph $G$ with eigfn $f$ associated with $\lambda_1$, define for each $v \in V$

$$C_v = \left\{ \{u, w\} \in E(G) \mid f(u) \leq f(v) < f(w) \right\}$$

and

$$\alpha = \min_v |C_v| \cdot \min \left\{ \sum_{u \mid f(u) \leq f(v)} d_u, \sum_{u \mid f(u) > f(v)} d_u \right\}^{-1}.$$

Then $\lambda_1 > 1 - \sqrt{1 - \alpha^2}$. 
LOWER BOUND ON $\lambda_1$

For a connected simple graph $G$, $h_G \geq \frac{2}{\text{vol } G}$.

From Cheeger's inequality, $2h_G \geq \lambda_1 > \frac{h_G^2}{2}$, we have

$$\lambda_1 > \frac{1}{2} \left( \frac{2}{\text{vol } G} \right)^2.$$  

As $\text{vol } G = 2|E(G)| \leq n(n - 1) \leq n^2$, we get a lower bound

$$\lambda_1 \geq \frac{2}{n^4}.$$