Laplacians and Eigenvalues

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The standard Laplacian

Graph G, vertices numbered $1, \ldots, n$. Degrees $d_1, \ldots, d_n$.
Adjacency matrix A, degree matrix $T = \mathrm{diag}(d_1, \ldots, d_n)$, incidence matrix E:

$$E(u,e) = \begin{cases} +1, & \text{if } u \text{ incident to } e = \{u,v\} \text{ with orientation } e : u \to v, \\ -1, & \text{if } u \text{ incident to } e = \{u,v\} \text{ with orientation } e : v \to u, \\ 0, & \text{otherwise.} \end{cases}$$

Standard Laplacian $L = EE^*$:

$$L(u,v) = \sum_e E(u,e)E(v,e) = \begin{cases} d_u, & \text{if } u = v, \\ -1, & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Thus also $L = T - A$.

The normalised Laplacian

For k-regular G, natural to consider also normalised Laplacian

$$\mathcal{L} \quad = \quad \frac{1}{k}L \quad = \quad I - \frac{1}{k}A.$$

In general, define *normalised incidence matrix* S:

$$S(u,e) = \begin{cases} +1/\sqrt{d_u}, & \text{if } u \text{ incident to } e = \{u,v\} \text{ with orientation } e: u \to v, \\ -1/\sqrt{d_u}, & \text{if } u \text{ incident to } e = \{u,v\} \text{ with orientation } e: v \to u, \\ 0, & \text{otherwise.} \end{cases}$$

Then define *normalised Laplacian* $\mathcal{L} = SS^*$:

$$\mathcal{L}(u,v) \quad = \quad \sum_e S(u,e)S(v,e) \quad = \quad \begin{cases} 1, & \text{if } u = v, \\ -1/\sqrt{d_u d_v}, & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Relation to standard Laplacian

Denote

$$T^{1/2} = \mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n}).$$

Since for all vertices u, v in G:

$$\mathcal{L}(u,v) \quad = \quad \frac{L(u,v)}{\sqrt{d_u d_v}},$$

one obtains representation

$$\mathcal{L} \quad = \quad T^{-1/2} L T^{-1/2} \quad = \quad I - T^{-1/2} A T^{-1/2}.$$

[Note that

$$LT^{-1/2}(u,v) = \sum_w L(u,w) T^{-1/2}(w,v) = L(u,v) \cdot \frac{1}{\sqrt{d_v}}$$

$$T^{-1/2}L(u,v) = \sum_w T^{-1/2}(u,w) L(w,v) = \frac{1}{\sqrt{d_u}} \cdot L(u,v).]$$

The Laplacian as operator

Consider an assignment of vertex potentials $g : V(G) \to \mathbb{R}$. Then:

$$Lg(u) = (T - A)g(u) = d_u \cdot g(u) - \sum_{v \sim u} g(v) = \sum_{v \sim u} (g(u) - g(v)).$$

For k-regular G, the normalised Laplacian yields:

$$\mathcal{L}g(u) = (I - \frac{1}{k}A)g(u) = g(u) - \frac{1}{k}\sum_{v \sim u} g(v) = \frac{1}{k}\sum_{v \sim u} (g(u) - g(v)).$$

For general G, the normalised Laplacian yields:

$$\begin{aligned}\mathcal{L}g(u) &= (I - T^{-1/2}AT^{-1/2})g(u) \\ &= g(u) - \sum_{v \sim u} \frac{g(v)}{\sqrt{d_u d_v}} \quad = \quad \frac{1}{\sqrt{d_u}} \sum_{v \sim u} (\frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}}).\end{aligned}$$

This leads to the notion of *normalised (harmonic) potentials*:

$$f(u) = T^{-1/2}g(u) = \frac{g(u)}{\sqrt{d_u}}.$$

Laplacian eigenvalues

Since $\mathcal{L}$ is symmetric, all its eigenvalues are real. Consider an eigenvalue λ with associated eigenvector g:

$$\begin{aligned}\mathcal{L}g = \lambda g &\iff (T^{-1/2}LT^{-1/2})g = \lambda g \\ &\iff L(T^{-1/2}g) = \lambda(T^{1/2}g) \\ &\iff Lf = \lambda Tf \\ &\iff (T^{-1}L)f = \lambda f.\end{aligned}$$

Thus g is eigenvector of $\mathcal{L}$ with eigenvalue λ
$\iff$ $f = T^{-1/2}g$ is eigenvector of $T^{-1}L$ with eigenvalue λ.
I.e. $\mathcal{L}$ has same spectrum as $T^{-1}L$, with "normalised" eigenvectors.

$$T^{-1}L = T^{-1}T - T^{-1}A = I - T^{-1}A,$$

$$T^{-1}L(u,v) = \begin{cases} 1, & \text{if } u = v, \\ -1/d_u, & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Rayleigh quotient

$$
\begin{aligned}
\frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} &= \frac{\langle g, T^{-1/2} L T^{-1/2} g \rangle}{\langle g, g \rangle} \\
&= \frac{\langle T^{1/2} f, T^{-1/2} L f \rangle}{\langle T^{1/2} f, T^{1/2} f \rangle} \\
&= \frac{\langle f, L f \rangle}{\langle f, T f \rangle} \\
&= \frac{\sum_u f(u) \sum_{v \sim u} (f(u) - f(v))}{\sum_u f(u) \cdot d_u f(u)} \\
&= \frac{\sum_{u \sim v} (f(u)^2 + f(v)^2 - 2 f(u) f(v))}{\sum_u f(u)^2 d_u} \\
&= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f(u)^2 d_u}
\end{aligned}
$$

Variational characterisations

Enumerate eigenvalues of $\mathcal{L}$ as $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$. Then:

$$\lambda_0 = \inf_g \frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} = \inf_f \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_u f(u)^2 d_u} = 0.$$

Clearly $\lambda_0 = 0$ has eigenvector $f \equiv 1$, i.e.
$g = T^{1/2}1 = (\sqrt{d_1}, \ldots, \sqrt{d_n})$.

The next eigenvalue $\lambda_G = \lambda_1$ is given by:

$$\lambda_1 = \inf_{g \perp T^{1/2}1} \frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} = \inf_{f \perp T1} \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_u f(u)^2 d_u}.$$

Note that

$$f \perp T1 \iff \sum_u f(u) d_u = 0.$$

Variational characterisations (cont'd)

The eigenvalue λ_1 can also be characterised as follows (exercise):

$$\lambda_1 = \inf_f \sup_t \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u (f(u)-t)^2 d_u}$$

and

$$\lambda_1 = \inf_f \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u (f(u)-\bar{f})^2 d_u},$$

where

$$\bar{f} = \frac{\sum_u f(u) d_u}{\sum_u d_u}.$$

Denoting $\sum_u d_u = \mathrm{vol}\, G$, one has yet another characterisation:

$$\lambda_1 = \mathrm{vol}\, G \cdot \inf_{f\neq 0} \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_{u:v}(f(u)-f(v))^2 d_u d_v},$$

where $\sum_{u:v}$ denotes summation over all unordered pairs of vertices u, v in G with possibly $u = v$.

Variational characterisations (cont'd)

In general,

$$\lambda_k = \inf_{f \perp TP_{k-1}} \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u f(u)^2 d_u}$$
$$= \inf_{f\neq 0} \sup_{g\in P_{k-1}} \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u (f(u)-g(u))^2 d_u},$$

where P_{k-1} denotes the subspace spanned by eigenvectors associated to eigenvalues $\lambda_0 \ldots \lambda_{k-1}$.

Finally,

$$\lambda_{n-1} = \sup_f \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u f(u)^2 d_u}.$$

Basic properties

Graph G with n vertices.

1. $\sum_i \lambda_i \leq n$, with equality iff G has no isolated vertices.
2. $\lambda_1 \leq n/(n-1)$ for $n \geq 2$, with equality iff G is complete.
   $\lambda_{n-1} \geq n/(n-1)$, unless G has isolated vertices.
3. $\lambda_1 \leq 1$, unless G is complete.
4. $\lambda_1 > 0$, if G is connected.
   More generally, if i is smallest index for which $\lambda_i > 0$, then G has exactly i connected components.
5. $\lambda_i \leq 2$ for all i, with $\lambda_{n-1} = 2$ iff G has a nontrivial bipartite component.
6. The spectrum of G is the union of the spectra of its connected components.

Bipartite graphs

The following are equivalent:

1. G is bipartite.
2. G has i connected components, and $\lambda_{(n-1)-j} = 2$ for $j = 1, \ldots, i$.
3. For each λ_i, also $2 - \lambda_i$ is eigenvalue of G.

A lower bound for λ_1

Diameter of graph = max shortest-path distance between any two vertices.

Theorem. For a connected graph G with diameter D,

$$\lambda_1 \geq \frac{1}{D \operatorname{vol} G}.$$

Proof.
Choose harmonic potential f associated to λ_1.
Choose vertex u_0 with $|f(u_0)| = \max$.
Since $f \perp T1$, there is some vertex v_0 s.th. $f(u_0)f(v_0) < 0$. Denote shortest path connecting u_0 and v_0 by P.

A lower bound for λ_1 (cont'd)

Then:

$$\begin{aligned}\lambda_1 &= \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_u f(u)^2 d_u} \\ &\geq \frac{\sum_{\{u,v\}\in P}(f(u)-f(v))^2}{f(u_0)^2\cdot\sum_u d_u} \\ &\geq \frac{\frac{1}{D}(f(u_0)-f(v_0))^2}{f(u_0)^2\cdot \mathrm{vol}\, G} \\ &\geq \frac{1}{D\,\mathrm{vol}\, G}.\end{aligned}$$

The next-to-last inequality follows by Cauchy-Schwartz:

$$\sum_{\{u,v\}\in P}(f(u)-f(v))^2 \geq \frac{1}{D}\left[\sum_{\{u,v\}\in P}(f(u)-f(v))\right]^2 = \frac{1}{D}(f(u_0)-f(v_0))^2.$$

Miscellaneous

Proposition. Let f be a harmonic potential associated to eigenvalue λ. Then for any vertex u,

$$\frac{1}{d_u}\sum_{v\sim u}(f(u)-f(v))=\lambda f(u).$$

Proof. Follows by comparing coefficients in $T^{-1}Lf=\lambda f$.

Theorem. For a k-regular graph with n vertices,

$$\max_{i>0}|1-\lambda_i|\geq\sqrt{\frac{n-k}{(n-1)k}}.$$

Weighted graphs

Given graph G with vertex set V, weight function $w : V \times V \to \mathbb{R}$ satisfying:

- $w(u,v) = w(v,u)$ for all u, v,
- $w(u,v) \geq 0$ for all u, v,
- $w(u,v) > 0$ only if $u \sim v$ in G.

Define:

- *degree* of vertex u: $d_u = \sum_v w(u,v)$.
- *volume* of graph G: $\mathrm{vol}\, G = \sum_u d_u$.

Laplacians of weighted graphs

Standard Laplacian:

$$L(u,v) = \begin{cases} d_u - w(u,u), & \text{if } u = v, \\ -w(u,v), & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$Lf(u) = \sum_{v \sim u} (f(u) - f(v)) w(u,v).$$

Normalised Laplacian: $\mathcal{L} = T^{-1/2} L T^{-1/2}$.

Thus,

$$\mathcal{L}(u,v) = \begin{cases} 1 - \frac{w(u,u)}{d_u}, & \text{if } u = v, \\ -\frac{w(u,v)}{\sqrt{d_u d_v}}, & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Variational characterisations

The previous characterisations still hold, *mutatis mutandis*. E.g.

$$\lambda_G := \lambda_1 = \inf_{g \perp T^{1/2}1} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle}$$

$$= \inf_{\sum f(u) d_u = 0} \frac{\sum_{u \sim v} (f(u) - f(v))^2 w(u,v)}{\sum_u f(u)^2 d_u}.$$

Contractions

A *contraction* of a graph is formed by identifying two distinct vertices u, v into a single vertex u^*. The weights of edges incident to u^* are determined as follows:

$$w(x, u^*) = w(x, u) + w(x, v)$$
$$w(u^*, u^*) = w(u, u) + w(v, v) + 2w(u, v).$$

Theorem. If graph H is formed by contractions from graph G, then

$$\lambda_G \leq \lambda_H.$$