Laplacians and Eigenvalues

Pekka Orponen

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The standard Laplacian

Graph $G$, vertices numbered $1, \ldots, n$. Degrees $d_1, \ldots, d_n$.

Adjacency matrix $A$, degree matrix $T = \text{diag}(d_1, \ldots, d_n)$, incidence matrix $E$:

$$E(u, e) = \begin{cases} +1, & \text{if } u \text{ incident to } e = \{u, v\} \text{ with orientation } e : u \to v, \\ -1, & \text{if } u \text{ incident to } e = \{u, v\} \text{ with orientation } e : v \to u, \\ 0, & \text{otherwise}. \end{cases}$$

Standard Laplacian $L = EE^*$:

$$L(u, v) = \sum_{e} E(u, e) E(v, e) = \begin{cases} d_u, & \text{if } u = v, \\ -1, & \text{if } u \neq v, u \sim v, \\ 0, & \text{otherwise}. \end{cases}$$

Thus also $L = T - A$. 
**The normalised Laplacian**

For $k$-regular $G$, natural to consider also normalised Laplacian

$$\mathcal{L} = \frac{1}{k} L = I - \frac{1}{k} A.$$ 

In general, define *normalised incidence matrix* $S$:

$$S(u, e) = \begin{cases} 
+1/\sqrt{d_u}, & \text{if } u \text{ incident to } e = \{u, v\} \text{ with orientation } e : u \to v, \\
-1/\sqrt{d_u}, & \text{if } u \text{ incident to } e = \{u, v\} \text{ with orientation } e : v \to u, \\
0, & \text{otherwise.}
\end{cases}$$

Then define *normalised Laplacian* $\mathcal{L} = SS^*$:

$$\mathcal{L}(u, v) = \sum_{e} S(u, e) S(v, e) = \begin{cases} 
1, & \text{if } u = v, \\
-1/\sqrt{d_u d_v}, & \text{if } u \neq v, u \sim v, \\
0, & \text{otherwise.}
\end{cases}$$
Relation to standard Laplacian

Denote

\[ T^{1/2} = \text{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n}). \]

Since for all vertices \( u, v \) in \( G \):

\[ L(u, v) = \frac{L(u, v)}{\sqrt{d_u d_v}}, \]

one obtains representation

\[ L = T^{-1/2}LT^{-1/2} = I - T^{-1/2}AT^{-1/2}. \]

[Note that

\[ LT^{-1/2}(u, v) = \sum_w L(u, w)T^{-1/2}(w, v) = L(u, v) \cdot \frac{1}{\sqrt{d_v}} \]

\[ T^{-1/2}L(u, v) = \sum_w T^{-1/2}(u, w)L(w, v) = \frac{1}{\sqrt{d_u}} \cdot L(u, v). \]
The Laplacian as operator

Consider an assignment of vertex potentials \( g : V(G) \rightarrow \mathbb{R} \). Then:

\[
Lg(u) = (T - A)g(u) = d_u \cdot g(u) - \sum_{v \sim u} g(v) = \sum_{v \sim u} (g(u) - g(v)).
\]

For \( k \)-regular \( G \), the normalised Laplacian yields:

\[
\mathcal{L}g(u) = (I - \frac{1}{k} A)g(u) = g(u) - \frac{1}{k} \sum_{v \sim u} g(v) = \frac{1}{k} \sum_{v \sim u} (g(u) - g(v)).
\]

For general \( G \), the normalised Laplacian yields:

\[
\mathcal{L}g(u) = (I - T^{-1/2} AT^{-1/2})g(u)
\]

\[
= g(u) - \sum_{v \sim u} \frac{g(v)}{\sqrt{d_u d_v}} = \frac{1}{\sqrt{d_u}} \sum_{v \sim u} \left( \frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}} \right).
\]

This leads to the notion of normalised (harmonic) potentials:

\[
f(u) = T^{-1/2} g(u) = \frac{g(u)}{\sqrt{d_u}}.
\]
Laplacian eigenvalues

Since $L$ is symmetric, all its eigenvalues are real. Consider an eigenvalue $\lambda$ with associated eigenvector $g$:

$$Lg = \lambda g \iff (T^{-1/2}LT^{-1/2})g = \lambda g$$

$$\iff L(T^{-1/2}g) = \lambda (T^{1/2}g)$$

$$\iff Lf = \lambda Tf$$

$$\iff (T^{-1}L)f = \lambda f.$$  

Thus $g$ is eigenvector of $L$ with eigenvalue $\lambda$

$$\iff f = T^{-1/2}g$$

is eigenvector of $T^{-1}L$ with eigenvalue $\lambda$.

I.e. $L$ has same spectrum as $T^{-1}L$, with “normalised” eigenvectors.

$$T^{-1}L = T^{-1}T - T^{-1}A = I - T^{-1}A,$$

$$T^{-1}L(u, \nu) = \begin{cases} 1, & \text{if } u = \nu, \\ -1/d_u, & \text{if } u \neq \nu, u \sim \nu, \\ 0, & \text{otherwise}. \end{cases}$$
Rayleigh quotient

\[
\frac{\langle g, Lg \rangle}{\langle g, g \rangle} = \frac{\langle g, T^{-1/2}LT^{-1/2}g \rangle}{\langle g, g \rangle}
\]

\[
= \frac{\langle T^{1/2}f, T^{-1/2}Lf \rangle}{\langle T^{1/2}f, T^{1/2}f \rangle}
\]

\[
= \frac{\langle f, Lf \rangle}{\langle f, Tf \rangle}
\]

\[
= \frac{\sum_u f(u) \sum_{v \sim u} (f(u) - f(v))}{\sum_u f(u) \cdot d_u f(u)}
\]

\[
= \frac{\sum_{u \sim v} (f(u)^2 + f(v)^2 - 2f(u)f(v))}{\sum_u f(u)^2 d_u}
\]

\[
= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f(u)^2 d_u}
\]
Variational characterisations

Enumerate eigenvalues of $L$ as $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$. Then:

$$\lambda_0 = \inf_{g} \frac{\langle g, Lg \rangle}{\langle g, g \rangle} = \inf_{f} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{u} f(u)^2 d_u} = 0.$$ 

Clearly $\lambda_0 = 0$ has eigenvector $f \equiv 1$, i.e. $g = T^{1/2} 1 = (\sqrt{d_1}, \ldots, \sqrt{d_n})$.

The next eigenvalue $\lambda_G = \lambda_1$ is given by:

$$\lambda_1 = \inf_{g \perp T^{1/2} 1} \frac{\langle g, Lg \rangle}{\langle g, g \rangle} = \inf_{f \perp T1} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{u} f(u)^2 d_u}.$$ 

Note that

$$f \perp T1 \iff \sum_{u} f(u) d_u = 0.$$
Variational characterisations (cont’d)

The eigenvalue $\lambda_1$ can also be characterised as follows (exercise):

$$\lambda_1 = \inf_f \sup_t \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u (f(u) - t)^2 d_u}$$

and

$$\lambda_1 = \inf_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u (f(u) - \bar{f})^2 d_u},$$

where

$$\bar{f} = \frac{\sum_u f(u) d_u}{\sum_u d_u}.$$  

Denoting $\sum_u d_u = \text{vol } G$, one has yet another characterisation:

$$\lambda_1 = \text{vol } G \cdot \inf_{f \neq 0} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{u:v} (f(u) - f(v))^2 d_u d_v},$$

where $\sum_{u:v}$ denotes summation over all unordered pairs of vertices $u$, $v$ in $G$ with possibly $u = v$.  

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Variational characterisations (cont’d)

In general,

$$\lambda_k = \inf_{f \perp TP_{k-1}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f(u)^2 du}$$

$$= \inf_{f \neq 0} \sup_{g \in P_{k-1}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u (f(u) - g(u))^2 du},$$

where $P_{k-1}$ denotes the subspace spanned by eigenvectors associated to eigenvalues $\lambda_0 \ldots \lambda_{k-1}$.

Finally,

$$\lambda_{n-1} = \sup_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f(u)^2 du}.$$
Basic properties

Graph $G$ with $n$ vertices.

1. $\sum \lambda_i \leq n$, with equality iff $G$ has no isolated vertices.
2. $\lambda_1 \leq n/(n-1)$ for $n \geq 2$, with equality iff $G$ is complete.
   $\lambda_{n-1} \geq n/(n-1)$, unless $G$ has isolated vertices.
3. $\lambda_1 \leq 1$, unless $G$ is complete.
4. $\lambda_1 > 0$, if $G$ is connected.
   More generally, if $i$ is smallest index for which $\lambda_i > 0$, then $G$ has exactly $i$ connected components.
5. $\lambda_i \leq 2$ for all $i$, with $\lambda_{n-1} = 2$ iff $G$ has a nontrivial bipartite component.
6. The spectrum of $G$ is the union of the spectra of its connected components.
Bipartite graphs

The following are equivalent:

1. $G$ is bipartite.

2. $G$ has $i$ connected components,
   and $\hat{\lambda}_{(n-1) - j} = 2$ for $j = 1, \ldots, i$.

3. For each $\lambda_i$, also $2 - \lambda_i$ is eigenvalue of $G$. 
A lower bound for \( \lambda_1 \)

**Diameter** of graph = max shortest-path distance between any two vertices.

**Theorem.** For a connected graph \( G \) with diameter \( D \),

\[
\lambda_1 \geq \frac{1}{D \ vol \ G}.
\]

**Proof.**

Choose harmonic potential \( f \) associated to \( \lambda_1 \).
Choose vertex \( u_0 \) with \( |f(u_0)| = \max \).
Since \( f \perp T1 \), there is some vertex \( v_0 \) s.th. \( f(u_0)f(v_0) < 0 \). Denote shortest path connecting \( u_0 \) and \( v_0 \) by \( P \).
A lower bound for $\lambda_1$ (cont’d)

Then:

$$
\lambda_1 = \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f(u)^2 d_u} \geq \frac{\sum \{u, v\} \in P (f(u) - f(v))^2}{f(u_0)^2 \cdot \sum u d_u} \geq \frac{1}{D} \frac{(f(u_0) - f(v_0))^2}{f(u_0)^2 \cdot \text{vol } G} \geq \frac{1}{D \text{ vol } G}.
$$

The next-to-last inequality follows by Cauchy-Schwartz:

$$
\sum \{u, v\} \in P (f(u) - f(v))^2 \geq \frac{1}{D} \left[ \sum \{u, v\} \in P (f(u) - f(v)) \right]^2 = \frac{1}{D} (f(u_0) - f(v_0))^2.
$$
Miscellaneous

**Proposition.** Let $f$ be a harmonic potential associated to eigenvalue $\lambda$. Then for any vertex $u$,

$$
\frac{1}{d_u} \sum_{v \sim u} (f(u) - f(v)) = \lambda f(u).
$$

*Proof.* Follows by comparing coefficients in $T^{-1}Lf = \lambda f$.

**Theorem.** For a $k$-regular graph with $n$ vertices,

$$
\max_{i>0} |1 - \lambda_i| \geq \sqrt{\frac{n-k}{(n-1)k}}.
$$
Weighted graphs

Given graph $G$ with vertex set $V$, weight function $w : V \times V \to \mathbb{R}$ satisfying:

- $w(u, v) = w(v, u)$ for all $u, v$,
- $w(u, v) \geq 0$ for all $u, v$,
- $w(u, v) > 0$ only if $u \sim v$ in $G$.

Define:

- **degree** of vertex $u$: $d_u = \sum_v w(u, v)$.
- **volume** of graph $G$: $\text{vol } G = \sum_u d_u$. 
Laplacians of weighted graphs

Standard Laplacian:

\[ L(u, v) = \begin{cases} 
  d_u - w(u, u), & \text{if } u = v, \\
  -w(u, v), & \text{if } u \neq v, u \sim v, \\
  0, & \text{otherwise.}
\end{cases} \]

Thus,

\[ Lf(u) = \sum_{v \sim u} (f(u) - f(v)) w(u, v). \]

Normalised Laplacian: \( L = T^{-1/2}LT^{-1/2} \).

Thus,

\[ L(u, v) = \begin{cases} 
  1 - \frac{w(u, u)}{d_u}, & \text{if } u = v, \\
  -\frac{w(u, v)}{\sqrt{d_u d_v}}, & \text{if } u \neq v, u \sim v, \\
  0, & \text{otherwise.}
\end{cases} \]
Variational characterisations

The previous characterisations still hold, *mutatis mutandis*. E.g.

\[ \lambda_G := \lambda_1 = \inf_{g \perp T^{1/2}} \frac{\langle g, Lg \rangle}{\langle g, g \rangle} \]

\[ = \inf \frac{\sum_{u \sim v} (f(u) - f(v))^2 w(u, v)}{\sum_u f(u)^2 d_u}. \]
Contractions

A contraction of a graph is formed by identifying two distinct vertices $u$, $v$ into a single vertex $u^*$. The weights of edges incident to $u^*$ are determined as follows:

\[ w(x, u^*) = w(x, u) + w(x, v) \]
\[ w(u^*, u^*) = w(u, u) + w(v, v) + 2w(u, v). \]

**Theorem.** If graph $H$ is formed by contractions from graph $G$, then

\[ \lambda_G \leq \lambda_H. \]