Vertex-partitions and the spectrum
(Biggs sec. 8)

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**Definitions**

colour-partition $V_i$: $V\Gamma = V_1 \cup V_2 \cup \ldots \cup V_l$, so that each $V_i (1 \leq i \leq l)$ contains no pair of adjacent vertices.

chromatic number $\nu(\Gamma)$ is the least natural number $l$ for which such partition is possible.

vertex-colouring: assignment of vertex colours with adjacent vertices having different colour. Vertex-colouring with $l$ colours gives rise to a colour partition with $l$ colour-classes.

A graph is $l$-critical if $\nu(\Gamma) = l$ and for all induced subgraphs $\Lambda \neq \Gamma$ we have $\nu(\Lambda) < l$. 
Lemma The chromatic number of a graph $X$ is the least integer $l$ such that there is a homomorphism from $X$ to $K_l$. (Codsil and Royle)

A $l$-critical graph cannot have a homomorphism to any proper subgraph, and hence must be its own core. This provides a wide class of cores, including all complete graphs and odd cycles. (Codsil and Royle)

The Four Colour Theorem Every planar graph has a vertex-colouring with $l = 4$ colours
Definitions (cont.)

Rayleigh quotient: \( R(X; z) = \langle z, Xz \rangle / \langle z, z \rangle \)

maximum and minimum eigenvalues of \( X \) \( \lambda_{max}(X) \) and \( \lambda_{min}(X) \)

\[ \lambda_{max}(X) \geq R(X; z) \geq \lambda_{min}(X) \]
for all \( z \neq 0 \).

Proof: \( U^t X U = D \). Substituting \( X = UDU^t \) and \( z = Uy \).

\[ R(X; y) = \langle y, Dy \rangle / \langle y, y \rangle = y^t Dy / y^t (U^t U)y = \sum_i \lambda_i |y_i|^2 / \sum_i |y_i|^2 \]

so the equations above hold.
Proposition 8.3

(1) If $\Lambda$ is an induced subgraph of $\Gamma$, then
$\lambda_{\text{max}}(\Lambda) \leq \lambda_{\text{max}}(\Gamma)$; $\lambda_{\text{min}}(\Lambda) \geq \lambda_{\text{min}}(\Gamma)$.

(2) If the greatest and least degrees among the vertices of $\Gamma$ are $k_{\text{max}}(\Gamma)$ and $k_{\text{min}}(\Gamma)$, and the average degree is $k_{\text{ave}}(\Gamma)$, then
$k_{\text{max}}(\Gamma) \geq \lambda_{\text{max}}(\Gamma) \geq k_{\text{ave}}(\Gamma) \geq k_{\text{min}}(\Gamma)$. 
**Lemma 8.4** Suppose $\Gamma$ is a graph with chromatic number $l \geq 2$. Then $\Gamma$ has a $l$-critical induced subgraph $\Lambda$, and every vertex of $\Lambda$ has degree at least $l - 1$ in $\Lambda$.

Proof: The set of induced subgraphs of $\Gamma$ is non-empty and contains graphs with $\nu$ of $l$ and graphs with $\nu$ of not $l$. Let $\Lambda$ be an induced subgraph with $\nu(\Lambda) = l$, and with minimal $|V\Lambda|$, then $\Lambda$ is $l$-critical. Since if $\Lambda$ were not $l$-critical, there would be a induced subgraph $\Upsilon \neq \Lambda$ of $\Lambda$ with $\nu(\Upsilon) \geq l$. But this means that $\Lambda$ is not minimal in size of the induced subgraphs of $\Gamma$ with $\nu(\Lambda) = l$. 
If $v \in \Lambda$, then $\langle V \Lambda \setminus v \rangle$ is an induced subgraph of $\Lambda$ and has a vertex-colouring with $l - 1$ colours. If the degree of $v$ in $\Lambda$ is less than $l - 1$, then we could extend this vertex-colouring to $\Lambda$, contradicting the fact that $\nu(\Lambda) = l$. Thus the degree of $v$ is at least $l - 1$. 
**Proposition 8.5** For any graph $\Gamma$ we have $\nu(\Gamma) \leq 1 + \lambda_{max}(\Gamma)$.

**Lemma 8.6** Let $X$ be a real symmetric matrix, partitioned in the form

$$A = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$$

, where $P$ and $Q$ are square symmetric matrices. Then $\lambda_{max}(X) + \lambda_{min}(X) \leq \lambda_{max}(P) + \lambda_{max}(R)$. 
**Corollary 8.7** Let $A$ be a real symmetric matrix, partitioned into $t^2$ submatrices $A_{ij}$ in such a way that the row and column partitions are the same; in other words, each diagonal sub-matrix $A_{ii}(1 \leq i \leq t)$ is square, then

$$\lambda_{max}(A) + (t - 1)\lambda_{min}(A) \leq \sum_{i=0}^{t} \lambda_{max}(A_{ii}).$$

**Theorem 8.8**

For any graph $\Gamma$, whose edge set is non-empty,

$$\nu(\Gamma) \geq 1 + \frac{\lambda_{max}(\Gamma)}{-\lambda_{min}(\Gamma)}.$$