# The tree-number and determinant expansions (Biggs 6-7)

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# Overview

- The *tree-number* $\kappa(\Gamma)$
- $\kappa(\Gamma)$ and the Laplacian matrix
- The $\sigma$ function
- *Elementary (sub)graphs*
- Coefficients of $\chi(\Gamma, \lambda)$ revisited
- The tree-number and forests

# The *tree-number*

**Definition:**
The number of spanning trees of a graph $\Gamma$ is its *tree-number*, denoted by $\kappa(\Gamma)$.

$\Gamma$ disconnected $\rightarrow \kappa(\Gamma) = 0$
If $\Gamma$ equals $K_n \rightarrow \kappa(\Gamma) = n^{n-2}$

# Laplacian matrix $Q$

Recall from section 4: Laplacian matrix $Q = DD^T$.

**Lemma:**
Let $D$ be the incidence matrix of a graph $\Gamma$, and let $Q$ be the Laplacian matrix. Then the adjugate of $Q$ is a multiple of $J$, where $J$ is the *all-ones* matrix.

Recall from linear algebra:

- Define minor $M_{ij}$ of $A$ as the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting row $i$ and column $j$ of $A$ and the cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.
- Then define the adjugate $adj(A)_{ij} := C_{ji}$.
- $A \; adj(A) = adj(A) \; A = det(A) \; I$

# ***Tree-number* [1]**

**Lemma:**
Every cofactor of $Q$ is equal to the *tree-number* of $\Gamma$, i.e. :

$$adj(Q) = \kappa(\Gamma)J$$

Recall from section 4:

$$Q = \Delta - A, \text{where } \Delta \text{ contains the degree of each vertex on the diagonal}$$

Thus, for the complete graph $K_n$:

$$Q = nI - J \rightarrow C_{ij} = n^{n-2}$$

# ***Tree-number*** [2]

**Proposition:**
The *tree-number* of a graph $\Gamma$ with n vertices is given by the formula

$$\kappa(\Gamma) = n^{-2} det(J + Q)$$

Defined in the results of section 4:
The *Laplacian Spectrum* of graph $\Gamma$ is the spectrum of its Laplacian matrix $Q = DD^T$ (eigenvalues).

**Corollary:**
Let $0 \leq \mu_1 \leq \ldots \leq \mu_{n-1}$ be the Laplacian spectrum of a graph $\Gamma$. Then:

$$\kappa(\Gamma) = \frac{\mu_1 \mu_2 \cdots \mu_{n-1}}{n}$$

# ***Tree-number*** **[3]**

If $\Gamma$ is connected and k-regular, and its spectrum is

$$\mathrm{Spec}\Gamma = \begin{pmatrix} k & \lambda_1 & \ldots & \lambda_{s-1} \\ 1 & m_1 & \ldots & m_{s-1} \end{pmatrix}$$

then

$$\kappa(\Gamma) = n^{-1} \prod_{r=1}^{s-1} (k - \lambda_r)^{m_r} = n^{-1} \chi'(\Gamma, k),$$

where $\chi'$ denotes the derivative of the characteristic polynomial $\chi$.

Application:

$$\kappa(L(\Gamma)) = 2^{m-n+1} k^{m-n} \kappa(\Gamma)$$

# $\sigma$ function

**Definition:**

$$\sigma(\Gamma, \mu) := det(\mu I - Q)$$

(characteristic function of the Laplacian matrix)

**Proposition:**

- If $\Gamma$ is disconnected, then the $\sigma$ function for $\Gamma$ is the product of the $\sigma$ functions for the components of $\Gamma$.
- If $\Gamma$ is a k-regular graph, then $\sigma(\Gamma, \mu) = (-1)^n \chi(\Gamma, k - \mu)$.
- If $\Gamma^c$ is the complement of $\Gamma$, and $\Gamma$ has $n$ vertices, then $\kappa(\Gamma) = n^{-2}\sigma(\Gamma^c, n)$. (the complementary graph has the same vertex set and the complementary set of edges, see results section 3)

# Determinant expansion

**Definition:**

An *elementary* graph is a simple graph, each component of which is regular and has degree 1 or 2 $\leftrightarrow$ each component is a single edge ($K_2$) or a cycle ($C_r$). A *spanning elementary subgraph* of $\Gamma$ is an elementary subgraph which contains all vertices of $\Gamma$.

**Proposition:**

$$det(A) = \sum sgn(\pi) a_{1,\pi 1} a_{2,\pi 2} \ldots a_{n,\pi n},$$

where the summation is over all permutations $\pi$ of the set {1,2,. . . n}.

$$det(A) = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation is over all spanning elementary subgraphs $\Lambda$ of $\Gamma$.
(Recall: $r(\Gamma) = n - c$, $s(\Gamma) = m - n + c$)

# Example

Consider the complete graph $K_4$. There are only 2 kinds of elementary subgraphs with four vertices: pairs of disjoint edges (r=2 and s=0) and 4-cycles (r=3 and s=1). There are three subgraphs of each kind so we have

$$det(A) = 3(-1)^2 2^0 + 3(-1)^3 2^1 = -3$$

# Characteristic polynomial revisited

Let

$$\chi(\Gamma,\lambda) = \lambda^n + c_1\lambda^{n-1}c_2\lambda^{n-2} + \ldots + c_n.$$

**Proposition:**
The coefficients of the characteristic polynomial are given by

$$(-1)^i c_i = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation is over all elementary subgraphs $\Lambda$ of $\Gamma$ with $i$ vertices.

# Previous values for $c_i$

Previously, we found out:

1. $c_1 = 0 \leftrightarrow$ There is no elementary subgraph with one vertex.
2. $-c_2 =$ is the number of edges of $\Gamma \leftrightarrow$ The number of elementary graphs with two vertices, $r = 1$, $s = 0$
3. $-c_3 =$ twice the number of triangles in $\Gamma \leftrightarrow$ The number of elementary graphs with three vertices times 2, $r = 2$, $s = 1$

Similar: The only elementary graphs with 4 vertices are the cycle graph $C_4$ and the graph having two disjoint edges. Result:

$$c_4 = \text{number of pairs of disjoint edges in } \Gamma$$
$$- \text{ number of 4-cycles in } \Gamma$$

$r_1 = 2$, $s_1 = 0$, $r_2 = 3$, $s_2 = 1$

# $\sigma$ function revisited [1]

Let

$$\sigma(\Gamma, \mu) = det(\mu I - Q) = \mu^n + q_1 \mu^{n-1} + \ldots + q_{n-1}\mu + q_n.$$

The $(-1)^i q_i$ is the sum of the principal minors of $Q$ which have $i$ rows and columns. One can show:

$$q_1 = -2|ET|, \quad q_{n-1} = (-1)^{n-1} n\kappa(\Gamma), \quad q_n = 0.$$

# $\sigma$ function revisited [2]

Let $D(X, Y)$ denote the submatrix of the incidence matrix $D$ of $\Gamma$ defined by the rows corresponding to vertices in $X$ and the columns corresponding to edges in $Y$. (see also Proposition 5.4)

**Lemma:**
Let $V_0$ denote the vertex-set of the subgraph $< Y >$. Then $D(X, Y)$ is invertible if and only if the following conditions are satisfied:

1. $X$ is a subset of $V_0$;

2. $< Y >$ contains no cycles;

3. $V_0 \backslash X$ contains precisely one vertex from each component of $< Y >$.

# $\sigma$ function revisited [3]

**Definition:**
A graph $\Phi$ whose co-rank is zero is a *forest*; it is the union of components each of which is a tree. We shall use the symbol $p(\Phi)$ to denote the product of the number of vertices in the components of $\Phi$.

**Theorem:**

$$(-1)^i q_i = \sum p(\Phi) \quad (1 \leq i \leq n),$$

where the summation is over all sub-forests $\Phi$ of $\Gamma$ which have $i$ edges.

# Tree-number revisited

**Corollary:**
The tree-number of a graph $\Gamma$ is given by the formula

$$\kappa(\Gamma) = n^{n-2} \sum p(\Phi)(-n)^{-|E\Phi|},$$

where the summation is over all forests $\Phi$ which are subgraphs of the complement of $\Gamma$.

# $\chi$ and forests

**Proposition:**
Let $\Gamma$ be a regular graph of degree $k$, and let $\chi^{(i)}$ $(0 \leq i \leq n)$ denote the $ith$ derivative of the characteristic polynomial of $\Gamma$. Then

$$\chi^{(i)}(\Gamma, k) = i! \sum p(\Phi),$$

where the summation is over all forests $\Phi$ which are subgraphs of $\Gamma$ with $|E\Phi| = n - i$.