Graph G:

- vertex set $V(G)$
- edge set $E(G)$
- incidence relation: a subset of $V(G) \times E(G)$
  (every edge is incident with either one or two vertices)

Strict graph: every edge is incident with two vertices and no two edges are incident with the same pair of vertices.

Degree of a vertex v: number of edges incident with v: $\deg(v)$

Subgraph $\langle S \rangle_G$: choose a subset S of $E(G)$ together with all the vertices in G incident with edges in S.

Induced subgraph $\langle U \rangle_G$: choose a subset U of $V(G)$ together with all the edges in G incident only with vertices in U.

Given G, $|V(G)| = n$, the adjacency matrix of G is the n x n matrix **A** (considered over the complex field) with:

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

A is a real symmetric matrix, tr(**A**) = 0.

We want to be vertex labeling independent, so are interested in properties of **A** invariant under permutations of rows and columns. In particular, spectral properties:

characteristic polynomial of **A**, $\det(\lambda\mathbf{I} - \mathbf{A})$, is preserved by rows and columns permutations.

A is real and symmetric, so all the roots of the characteristic polynomial are real, so all of them are real eigenvalues of **A**:

if λ_i is a root of det(λ**I** - **A**), then there exists x_i: **A**$x_i = \lambda_i\, x_i$

Diagonalization: **A`** = **B**$^{-1}$**AB**, where **B** is orthogonal, **A`** is diagonal.

What we have on the diagonal of **A`**: all the characteristic roots, with their multiplicities.

Multiplicity of λ_i as a root of det(λ**I** - **A**) is equal to the dimension of the eigenspace corresponding to λ_i.

The spectrum of G: set of the eigenvalues of **A**(G), with their multiplicities:

$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$, or

$\lambda_0 \quad > \quad \lambda_1 \quad > \ldots > \quad \lambda_{s-1}$

$m(\lambda_0) \quad m(\lambda_1) \qquad\qquad m(\lambda_{s-1})$

Sometimes it is easy to compute spectra directly: K_n, $K_{n,m}$, …

The characteristic polynomial of G:

$$\chi(G, \lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_n$$

Proposition 1

(1) $c_1 = 0$;
(2) $-c_2 = |E(G)|$
(3) $-c_3 = 2\ \#(\text{triangles in } G)$

Thus, we see that $\chi(G, \lambda)$ contains valuable graphical information.

Adjacency algebra $A(G)$:

algebra of polynomials in **A** with complex coefficients, under the standard matrix operations.

The Cayley-Hamilton theorem gives us:

$\chi(G, \mathbf{A}(G)) = 0$

So, $\dim A(G) \leq n$.

Every element of $A(G)$ is a linear combination of powers of **A**, so let's take a closer look at those matrices.

A walk of length l in G between v_i and v_j:

$v_i = u_0, u_1, \ldots, u_l = v_j$,

where $u_{t\text{-}1}$ and u_t are adjacent for $t = 1, ..., l$.

<u>Lemma 2</u>

The number of walks of length l in G from v_i to v_j is $\mathbf{A}^{l}_{(ij)}$.

(Straightforward induction.)

G is connected:

every pair of vertices is connected by a walk.

Distance between v_i and v_j is the length of the shortest walk connecting them.

For a connected graph G, the maximum distance between its vertices is called the diameter of G. Will denote it $d(G)$ or simply d.

<u>Proposition 3</u>

Let G be a connected graph with adjacency algebra $A(G)$ and diameter d. Then $\dim(A(G)) \geq d + 1$.

If G has s distinct eigenvalues, then its minimum polynomial μ has degree s.
(Consider the corresponding diagonal matrix.)

Then, by virtue of the previous proposition, we have:

$d + 1 \leq \dim(A(G)) \leq \deg(\mu) = s.$

So, we obtained a lower bound on the number of distinct eigenvalues of a connected graph.

Reduction formula for $\chi(G, \lambda)$:

Assume $\deg(v_1) = 1$ (v_1 is a leaf), and v_2 is the vertex adjacent to v_1. Let G_1 be the induced subgraph on $G \setminus \{v_1\}$, and G_{12} the induced subgraph on $G \setminus \{v_1, v_2\}$. Then:

$$\chi(G, \lambda) = \lambda\, \chi(G_1, \lambda) - \chi(G_{12}, \lambda)$$

This can be used for computing characteristic polynomial of any tree.

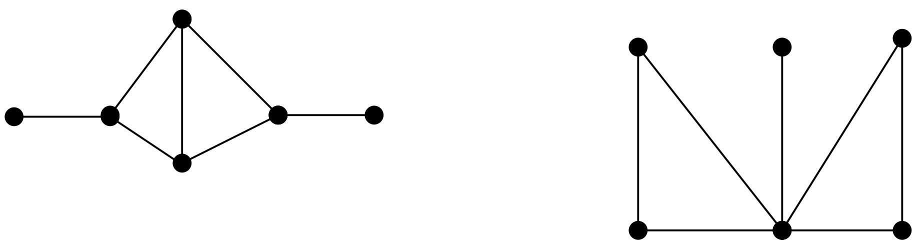
For P_n, the path graph on $n > 2$ vertices, we have:

$$\chi(P_n, \lambda) = \lambda\, \chi(P_{n-1}, \lambda) - \chi(P_{n-2}, \lambda)$$

Thus, $\chi(P_n, \lambda) = U_n(\lambda/2)$, where U_n is the Chebyshev polynomial of the second kind.

The spectrum of a bipartite graph is always symmetric with respect to 0.

Cospectral graphs: non-isomorphic graphs with the same characteristic polynomial.
(Predictably, characteristic polynomials do not contain <u>all</u> the graphical information.)



By Lemma 2, the total number of closed walks of length l is equal to $\mathrm{tr}(\mathbf{A}^l)$, that is, the sum of the eigenvalues of $\mathbf{A}^l$.

$\mathrm{tr}(\mathbf{A}^l) = \sum \lambda_i^{\,l}$

Then we immediately have that the sum of the squares of the eigenvalues is twice the number of edges, and the sum of cubes is six times the number of triangles.

This can be used from upper bounding the eigenvalues.

Let $|V(G)| = n$, $|E(G)| = m$.

We know that $\sum \lambda_i = 0$ and $\sum \lambda_i^{\,2} = 2m$. Using the quadratic vs. arithmetic means inequality, we obtain for the largest eigenvalue λ_0:

$\lambda_0 \le (2m\,(n-1)\,/\,n)^{1/2}$

Regular graphs.

G is k-regular if $\deg(v) = k$ for any vertex v of G.

<u>Proposition 4</u>

If G is k-regular, then

(1) k is an eigenvalue of G;
(2) if G is connected, then the multiplicity of k is 1;
(3) for any other eigenvalue λ, we have $|\lambda| \leq k$.

Let **J** denote the all 1's matrix. If **A** is the adjacency matrix of a k-regular graph, we have

$\mathbf{AJ} = \mathbf{JA} = k\mathbf{J}$

<u>Proposition 5</u>

The matrix **J** belongs to the adjacency algebra $A(G)$ iff G is a regular connected graph.

<u>Corollary 6</u>

Let G be a k-regular connected graph with n vertices, and let the distinct eigenvalues of G be $k > \lambda_1 > \ldots > \lambda_{s-1}$. Then if $q(\lambda) = \prod (\lambda - \lambda_i)$, over $i = 1, \ldots, s-1$, we have

$\mathbf{J} = (n / q(k))\, q(\mathbf{A})$.

Another special type of graphs: *circulant* graphs.

We call an n x n matrix **S** circulant if

$s_{ij} = s_{1,\, j-i+1 \text{ (mod } n)}$

(Row i is obtained from the first row by a cyclic shift of (i – 1) steps, **S** is fully determined by its first row.)

Denote by **W** the circulant matrix with the first row [0, 1, 0, …, 0].

We want to show that any circulant matrix **S** can be expressed as a linear combination of powers of **W**.

Note that for any matrix $\mathbf{X}$, $(\mathbf{XW})_{ij} = \mathbf{X}_{i,j\text{-}1}$, so multiplying by $\mathbf{W}$ shifts every row of $\mathbf{X}$ cyclically to the right by one step.

$\mathbf{W}^2$ is given by [0, 0, 1, 0, …, 0], and so on.

Thus, we can express $\mathbf{S}$ as:

$$\mathbf{S} = \sum s_i \, \mathbf{W}^{i\text{-}1}, \; i = 1, \ldots, n$$

since both of them are circulant and with the same first row.

Eigenvalues of **W** are the n-th roots of unity: 1, ω, ω^2, …, ω^{n-1}, where $\omega = \exp(2\pi i/n)$.
(We know $\lambda^n = 1$, and for any ω^r, we can set $x_1 = 1$ and compute/construct the entire eigenvector.)

It immediately gives us that the eigenvalues of **S** are:

$$\lambda_r = \sum s_i \, \omega^{r(i-1)}, \quad i = 1, \dots, n$$

where $r = 0, \dots, n-1$.

(Any eigenvector x of **W** is also an eigenvector of **S**.)

Thus, if G is a circulant graph with the first row of **A** being $[0, a_2, \dots, a_n]$, the eigenvalues of G are:

$$\lambda_r = \sum a_i \, \omega^{r(i-1)}, \quad i = 2, \dots, n$$

Some of these expressions may be equal, so multiplicities may be greater than 1.

Examples of circulant graphs:

(1) K_n: [0, 1, …, 1]

(2) Cycle graphs C_n: [0, 1, 0, …0, 1]

(3) Hyperoctahedral graphs H_s, obtained by removing s disjoint edges from K_{2s}:
[0, 1, …, 1, 0, 1, …, 1] (zeros are in the first and (s+1)st positions)

Line graphs.

Given G, we construct its line graph $L(G)$ by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$ whenever the corresponding edges in G are incident.

Let $V(G) = \{v_1, \ldots, v_n\}$, and $E(G) = \{e_1, \ldots, e_m\}$.

We define "almost incidence" matrix $\mathbf{X}(G)$, n x m, as follows:

$$\mathbf{X}_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } e_j \text{ are incident} \\ 0, \text{ otherwise} \end{cases}$$

<u>Lemma 7</u>

Let $\mathbf{A}$ denote the adjacency matrix of G and $\mathbf{A}_L$ the adjacency matrix of $L(G)$. Then:

(1) $\mathbf{X}^T\mathbf{X} = \mathbf{A}_L + 2\mathbf{I}_m$;

(2) If G is k-regular, then $\mathbf{X}\mathbf{X}^T = \mathbf{A} + k\mathbf{I}_n$.

(Just write down what $(\mathbf{X}^T\mathbf{X})_{ij}$ and $(\mathbf{X}\mathbf{X}^T)_{ij}$ are.)

<u>Proposition 8</u>

All eigenvalues of a line graph are not less than –2.

($\mathbf{X}^T\mathbf{X}$ is non-negative definite, so all its eigenvalues are non-negative, and we use the result of part (1) of Lemma 7.)

There are graphs, which are not line graphs, but all their eigenvalues are not less than –2.

There is a complete characterization of the graphs whose all eigenvalues are not less than –2.

If G is k-regular, then its line graph $L(G)$ is $(2k - 2)$-regular, an obvious connection between the maximum eigenvalues. In fact, there is a simple connection between the entire spectra of G and $L(G)$.

Theorem 9

If G is k-regular, $|V(G)| = n$, $|E(G)| = m = nk/2$, then we have

$$\chi(L(G), \lambda) = (\lambda + 2)^{m-n}\, \chi(G, \lambda + 2 - k)$$

So, we see that for $L(G)$, (-2) has multiplicity $(m - n)$ and eigenvalues $(k - 2 + \lambda_i)$ correspond to eigenvalues λ_i of G, with the same multiplicities.