Random Walks on Infinite Networks

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Recurrence and transience

Consider random walk on an infinite (but locally finite) graph G, starting from node a.

Denote: $p_{\mathit{esc}} = \Pr($walk starting at a never returns to $a)$.

$$p_{\mathit{esc}} = 0 \quad \Rightarrow \quad \text{walk is } \textit{recurrent}$$
$$p_{\mathit{esc}} > 0 \quad \Rightarrow \quad \text{walk is } \textit{transient}$$

Note: A recurrent walk visits every node x in G infinitely often, assuming $\Pr(a \rightsquigarrow x) > 0$.

Pólya's theorems for d-dimensional lattices:

$d = 1, 2$: random walk on $\mathbb{Z}^d$ recurrent

$d \geq 3$: random walk on $\mathbb{Z}^d$ transient

(For $d = 3$, $p_{\mathit{esc}} \approx 0.66$.)

Connection to electric networks

To analyse $p_{\textsf{esc}}$, consider graph $G^{(r)}$ consisting of nodes in G at most distance r away from a. Denote $\partial G^{(r)} =$ nodes at *exactly* distance r from a. Denoting

$$p^{(r)}_{\textsf{esc}} = \Pr(\text{walk starting at } a \text{ hits } \partial G^{(r)} \text{ before returning to } a),$$

we have $p_{\textsf{esc}} = \lim_{r\to\infty} p^{(r)}_{\textsf{esc}}$.

Analysis technique: consider unit resistor network obtained by setting a at high potential and grounding $\partial G^{(r)}$. Compute the effective conductance/resistance ($C^{(r)}_{\textsf{eff}}/R^{(r)}_{\textsf{eff}}$) between a and $\partial G^{(r)}$. Then (Section 1.3.4):

$$p^{(r)}_{\textsf{esc}} = \frac{C^{(r)}_{\textsf{eff}}}{C_a} = \frac{1}{(\deg a)\cdot R^{(r)}_{\textsf{eff}}}.$$

Background review

Consider reversible random walk on finite graph G with transition probabilities p_{xy} and stationary distribution π_x.

Define resistor network on G by:

$$\begin{aligned} C_x &\propto \pi_x, \\ C_{xy} &\propto \pi_x p_{xy} = \pi_y p_{yx}. \end{aligned}$$

Fix nodes a, b in G. Consider

$e_x =$ expected number of visits to node x by random walk starting at a, before hitting b.

Background (cont'd)

Then $v_x = \frac{e_x}{\pi_x} \propto \frac{e_x}{C_x}$ is harmonic w.r.t. G, p:

$$\begin{aligned}
\sum_{y \sim x} p_{xy} v_y &= \sum_{y \sim x} p_{xy} \frac{e_y}{\pi_y} \\
&= \sum_{y \sim x} p_{yx} \frac{\pi_y}{\pi_x} \frac{e_y}{\pi_y} \quad = \quad \frac{1}{\pi_x} \sum_{y \sim x} e_y p_{yx} \\
&= \frac{e_x}{\pi_x} \quad = \quad v_x.
\end{aligned}$$

Thus v_x is the *unique* harmonic assignment with $v_a = \frac{e_a}{\pi_a}$, $v_b = 0$; i.e. the v_x correspond to the voltages induced in the network by the given assignments at a and b.

Up to scaling, the same holds for any voltages $v_x = e_x / C_x$, where $C_x \propto \pi_x$.

Background (cont'd)

The currents induced by the voltages v_x are:

$$i_{xy} \quad = \quad (v_x - v_y)C_{xy} \quad = \quad (\frac{e_x}{C_x} - \frac{e_y}{C_y})C_{xy} \quad = \quad e_x p_{xy} - e_y p_{xy}.$$

In particular,

$$i_a = \sum_{y \sim a} i_{ay} = 1,$$

since the random walk started at a will eventually be absorbed at b.

If for a given resistor network one scales voltage at a from e_a/C_a to 1, then current at a is scaled from 1 to C_a/e_a.

Background (cont'd)

The effective resistance & conductance between a and b are:
$R_{\text{eff}} = v_a/i_a = e_a/C_a, \quad C_{\text{eff}} = i_a/v_a = C_a/e_a$.

When $v_a = 1$, voltages v_y correspond to probabilities of random walk starting at y hitting a before b, and so:

$$\begin{aligned} C_{\text{eff}} &= i_a = \sum_{y \sim a} (v_a - v_y) C_{ay} = \sum_{y \sim a} (v_a - v_y) \frac{C_{ay}}{C_a} C_a \\ &= C_a \sum_{y \sim a} (1 - v_y) p_{ay} = C_a (1 - \sum_{y \sim a} p_{ay} v_y) \\ &= C_a p_{\text{esc}}. \end{aligned}$$

Thus, one obtains the simple formula: $p_{\text{esc}} = \frac{C_{\text{eff}}}{C_a}$.