Random walks on finite networks

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Overview

- Short review of recent electric network models
- Model of electric networks with arbitrary resistors
- Markov chains for such networks
- Interpretation of voltage
- Interpretation of current

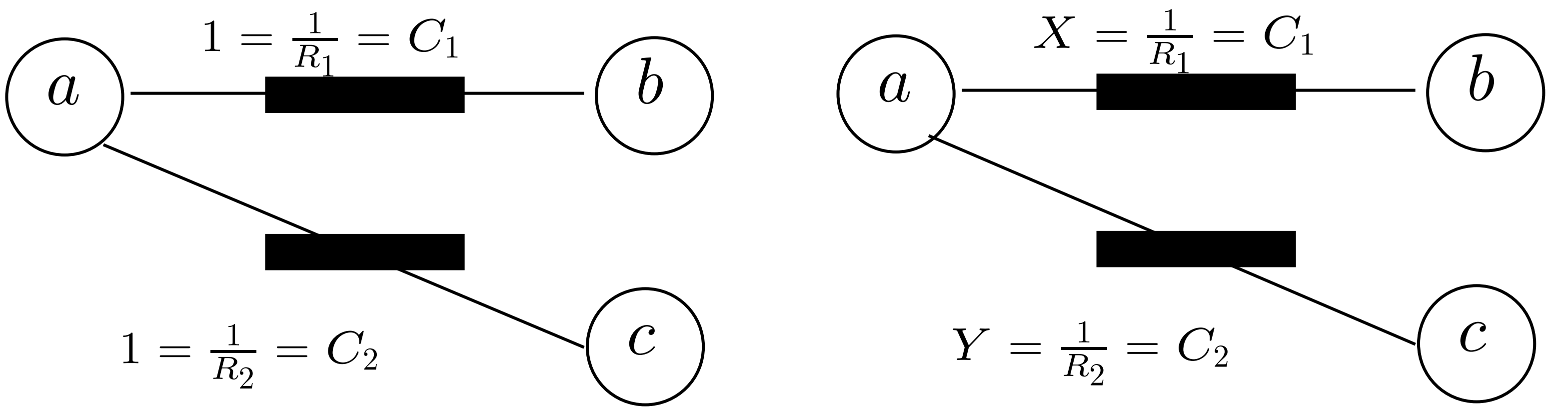
Review

- Random Walks and harmonic functions in one and two dimensions
- Uniqueness and Maximum Principle in one and two dimensions
- Four ways of finding the harmonic function ($\equiv$ solution to the Dirichlet problem):

  1. Monte Carlo method
  2. Method of relaxations
  3. Linear equations
  4. Markov chains

$\rightarrow$ So far, the model for electric networks only considered unit resistor values!

Network Model



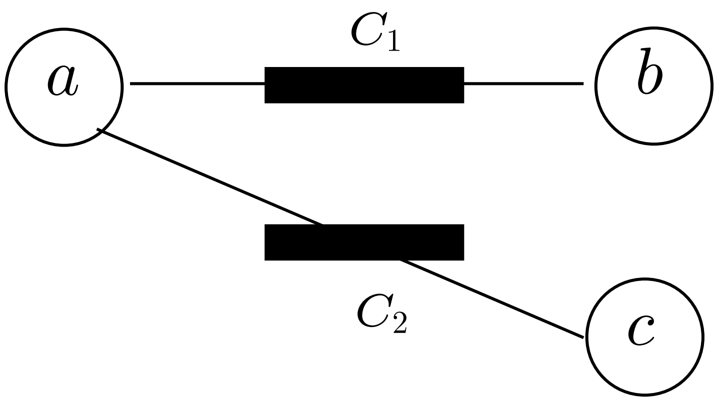
→ Rather than considering the resistor values R_{xy}, their reciprocal, the *conductance* C_{xy} is used.

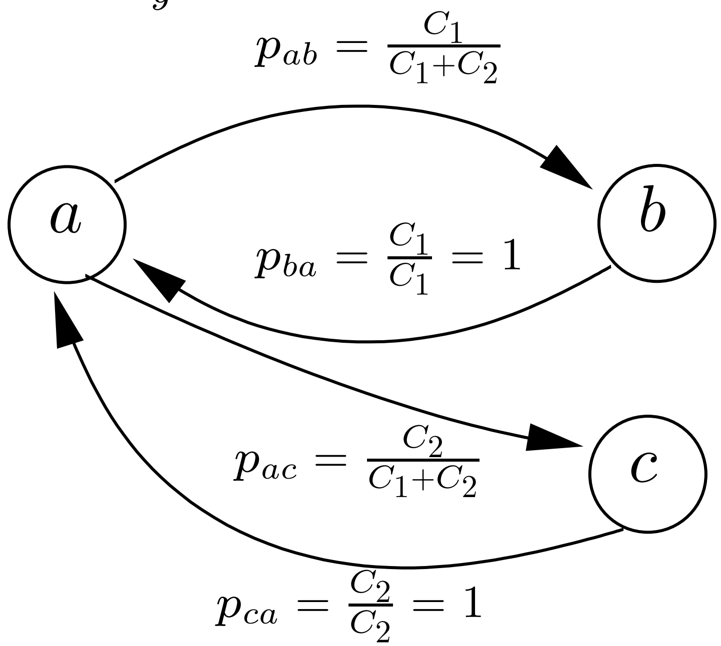
→ We consider an electric network to be a connected, weighted, undirected graph.

Random Walk ($:=$ Markov chain Model)

Definition: We define a *random walk* on a graph G modeling a resistor network to be a Markov chain with transition probabilities P_{xy}:

$$P_{xy} := \frac{C_{xy}}{C_x} \qquad C_x := \sum_y C_{xy}$$





Terminology

Definition: A Markov chain in which it is possible to reach every state from any other state is called *ergodic*.

Lemma: For an ergodic Markov chain, there is a unique probability vector w that is a fixed vector for P (left eigenvector with eigenvalue 1), i.e. it holds that $wP = w$. For our random walk on the resistor network:

$$w_x = \frac{C_x}{C} \qquad C = \sum_x C_x$$

Definition: An ergodic Markov chain for which the following holds is called *reversible*:

$$w_x * P_{xy} = w_y * P_{yx}$$

Lemma: If P is any reversible ergodic Markov chain, then P is the transition matrix for a random walk on an electric network with $C_{xy} := w_x * P_{xy}$.

Special case: $\forall x, y : C_{xy} := c$ (*simple random walk*)

Probabilistic Interpretation of Voltage (1/3)

- Let G be a network of resistors. Like before, we associate a voltage v_x to each node x and a current i_{xy} to each edge (x, y). Let $v_a = 1$ and $v_b = 0$.
- The following two laws are valid for "real" voltage and current and therefore have to be considered here, too:

Ohm's Law:

$$i_{xy} = \frac{v_x - v_y}{R_{xy}} = (v_x - v_y)C_{xy} \Rightarrow i_{xy} = -i_{yx}$$

Kirchhoff's Law:

$$\sum_y i_{xy} = 0$$

$$\Longrightarrow v_x = \sum_y \frac{C_{xy}}{C_x} v_y \Longrightarrow \text{Voltage } v_x \text{ is harmonic over all points } x \neq a, b$$

Probabilistic Interpretation of Voltage (2/3)

Proof:
Ohm & Kirchhoff $\Rightarrow$

$$\begin{aligned} \sum_y (v_x - v_y) C_{xy} &= 0 \\ \Rightarrow v_x = \sum_y \frac{C_{xy}}{C_x} v_y &= \sum_y P_{xy} v_y \qquad x \neq a, b \end{aligned}$$

$\Rightarrow v_x$ harmonic for P ($P v_x = v_x$) for all $x \neq a, b$

Probabilistic Interpretation of Voltage (3/3)

- Let h_x be the probability that starting at state x, the Markov chain/the random walker given by P (recall: $P_{xy} := \frac{C_{xy}}{C_x}$) reaches first state a before reaching b.
- Then h_x harmonic at all points $x \neq a, b$, $v_a = h_a = 1$ and $v_b = h_b = 0$.
- Modifying P to $\overline{P}$ by defining a and b to be absorbing states it follows by the uniqueness principle that $h_x = v_x$ and both are solutions to the Dirichlet problem.

Probabilistic Interpretation of Current (1/2)

- Naive idea: Assume that (electrically charged) particles enter the network at point/node a and traverse edges until they eventually reach point b and leave the network.

- Following the course of a single particle, we regard the current i_{xy} to be the expected number of edge traversals $x \to y$ (reverse traversals are negatives).

- The particle/random walker starts at a and keeps going in the event it returns to this point.

Probabilistic Interpretation of Current (2/2)

- Let u_x be the expected number of visits to state x before stating state b. Then one can show (using the reversibility of P and $u_x = \sum_y u_y P_{yx}$):

$$\frac{u_x}{C_x} = \sum_y P_{xy} \frac{u_y}{C_y} = v_x$$

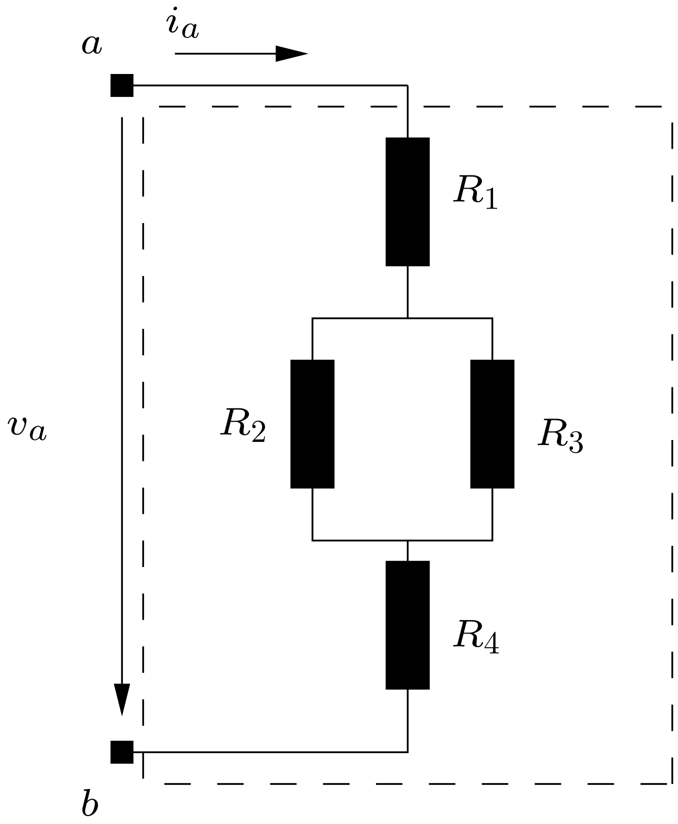
  The last equation holds because the left side function is harmonic for $x \neq a, b$ and has the same boundary values as v_x.

- Ohm's law implies:

$$i_{xy} = u_x P_{xy} - u_y P_{yx}$$

- However, the current i_{xy} is only proportional to the current flowing when a unit voltage is applied $\rightarrow$ the currents i_{xy} have to be normalized such that $\sum_y i_{ay} = \sum_y i_{yb} = 1$.

Effective Resistance / Escape Probability (1/2)



$$R_{eff} := \frac{v_a}{i_a} = R_1 + \frac{R_2 R_3}{R_2 + R_3} + R_4 = \frac{1}{C_{eff}}$$

Let $v_a = 1$ and let p_{esc} be the probability that the random walker starting at a reaches b before returning to a. Then:

$$p_{esc} = \frac{C_{eff}}{C_a}$$

Escape Probability (2/2)

Proof:

$$
\begin{aligned}
\frac{v_a}{i_a} &= \frac{1}{C_{eff}} \\
\Rightarrow C_{eff} &= i_a \qquad \text{for } v_a = 1 \\
i_a &= \sum_y (1 - v_y) C_{ay} = \sum_y C_{ay} - v_y \frac{C_{ay}}{C_a} C_a \\
&= C_a (1 - \sum_y P_{ay} v_y) \\
\Rightarrow i_a &= C_a p_{esc} \\
\Rightarrow p_{esc} &= \frac{C_{eff}}{C_a}
\end{aligned}
$$

End

Thank you for your attention. . .

- <Questions? / Discussion>
- <Break>
- <Exercises>