Problems to sections 6 and 7 of “Algebraic Graph Theory” by N. Biggs

1. [page 42, problem 6a]
   If \( \Gamma \) is a connected \( k \)-regular graph with \( n \) vertices, show using Corollary 6.5 and the arithmetic-geometric mean inequality:
   \[
   \kappa(\Gamma) \leq \frac{1}{n} \left( \frac{nk}{n-1} \right)^{n-1}
   \]
   with equality if and only if \( \Gamma = K_n \).

2. [page 49, problem 7b]
   The characteristic polynomial of a tree: Suppose that \( \sum c_i \lambda^{n-i} \) is the characteristic polynomial of a tree with \( n \) vertices. Show that the odd coefficients \( c_{2r+1} \) are zero, and the even coefficients \( c_{2r} \) are given by the rule that \( (-1)^r c_{2r} \) is the number of ways of choosing \( r \) disjoint edges in the tree.

3. [first part of page 49, problem 7d]
   The \( \sigma \) function of a star graph: A star graph is a complete bipartite graph \( K_{1,b} \). For such a graph we can calculate \( \sigma \) explicitly from the formula of Theorem 7.5. Show that
   \[
   \sigma(K_{1,b}\mu) = \mu(\mu - b - 1)(\mu - 1)^{b-1}
   \]

4. Let \( K_{n,m} \) be the complete bipartite graph (cmp. problems to sections 2 and 3). Calculate the number of elementary subgraphs of \( K_{n,m} \).