# T-79.5501 Cryptology

#### Lecture 8 (April 8, 2008):

- Wiener's Low decryption Exponent Attack, Sec 5.7.3

# **RSA Cryptosystem**

n = pq where p and q are two different large primes  $\phi(n) = (p-1)(q-1)$ 

*a* decryption exponent (private)

*b* encryption exponent (public)

 $ab \equiv 1 \pmod{\phi(n)}$ 

RSA operation:

 $(m^b)^a \equiv m \,(\mathrm{mod}\,n)$ 

for all m,  $0 \le m < n$ .

Wiener's result: It is insecure to select *a* shorter than about  $\frac{1}{4}$  of the length of *n*.

# **RSA Equation**

$$a\mathbf{b} - k \phi(n) = 1$$

for some k where only b is known.

Additional information: pq = n is known and q

$$n > \phi(n) = (p-1)(q-1) = pq - p - q + 1 \ge n - 3\sqrt{n}$$

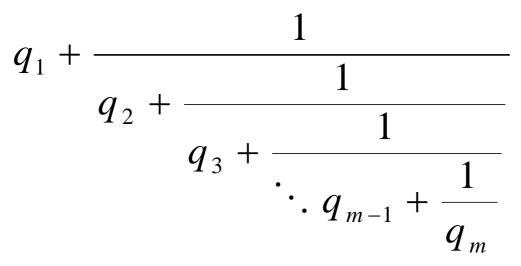
Also we know that  $a, b < \phi(n)$ , hence k < a.

Wiener (1989) showed how to exploit this information to solve for a and all other parameters k, p and q, if a is sufficiently small.

Wiener's method is based on continued fractions.

### **Continued Fractions**

Every rational number t has a unique representation as a finite chain of fractions



and we denote  $t = [q_1 q_2 q_3 \dots q_{m-1} q_m]$ . The rational number  $t_j = [q_1 q_2 q_3 \dots q_j]$  is called the *j*<sup>th</sup> convergent of *t*. For t = u/v, just run the Euclidean algorithm to find the  $q_i$ , i = 1, 2, ..., m.

#### **Convergent Lemma**

Theorem 5.14 Suppose that gcd(u,v) = gcd(c,d) = 1 and

$$\left|\frac{u}{v}-\frac{c}{d}\right| < \frac{1}{2d^2}.$$

Then c/d is one of the convergents of the continued fraction expansion of u/v.

Recall the RSA problem:  $ab - k\phi(n) = 1$ Write it as:

$$\frac{b}{\phi(n)} - \frac{k}{a} = \frac{1}{a\phi(n)}$$

Then, if  $2a < \phi(n)$ , then k/a is a convergent of  $b/\phi(n)$ .

#### Wiener's Theorem

If in RSA cryptosystem

$$a < \frac{1}{3} \sqrt[4]{n},$$

that is, the length of the private exponent a is less than about one forth of the length of n, then a can be computed in polynomial time with respect to the length of n.

Proof. First we show that k/a can be computed as a convergent of b/n, based on Euclidean algorithm, which is polynomial time. To see this, we estimate:

$$\left|\frac{b}{n}-\frac{k}{a}\right| = \left|\frac{ab-kn}{an}\right| = \left|\frac{1+k\phi(n)-kn}{an}\right| \le \frac{3k}{a\sqrt{n}} < \frac{3}{\sqrt{n}} < \frac{1}{2a^2}.$$

# Wiener's Algorithm

Then the convergents  $c_j/d_j = [q_1 q_2 q_3 \dots q_j]$  of b/n are computed. For the correct convergent  $k/a = c_j/d_j$  we have

$$bd_j - c_j \phi(n) = 1.$$

For each convergent one computes

$$n' = (d_j b - 1) / c_j$$

and checks if  $n' = \phi(n)$ . Note that  $p + q = n - \phi(n) + 1$ . Then if  $n' = \phi(n)$ , the equation

$$x^2 - (n - n' + 1)x + n = 0$$

has two positive integer solutions p and q.

#### EXAMPLE

Stinson, Problem 5.32: Suppose that n = 317940011 and b = 77537081 in the *RSA Cryptosystem*. Using Wiener's Algorithm, attempt to factor n. If you succeed, determine the secret exponent a and  $\phi(n)$ .

Solution: Running Wiener's algorithm we get:

| j | $r_j$     | $q_j$ | $c_j$ | $d_j$ | n'            |
|---|-----------|-------|-------|-------|---------------|
| 0 | 77537081  | 0     | 1     | 0     | -             |
| 1 | 317940011 | 0     | 0     | 1     | -             |
| 2 | 77537081  | 4     | 1     | 4     | 310148323     |
| 3 | 7791687   | 9     | 9     | 37    | 318763555.111 |
| 4 | 7411898   | 1     | 10    | 41    | 317902032     |
| 5 | 379789    | 19    | 199   | 816   | 317940995.452 |
| 6 | 195907    | 1     | 209   | 857   |               |
| : | :         | :     | ÷     | ÷     | :             |

j = 2 no solution, since n' is odd ( $\phi(n)$  is divisible by 4).

j = 3 no solution, since n' is not integer.

j = 4 looks promising. Substitute n = 317940011 and n' = 317902032 to equation  $x^2 - (n - n' + 1)x + n = 0$  and get

 $x^2 - 37980x + 317940011 = 0,$ 

from where we get solutions for p and q, which are  $x = 18990 \pm 6533$ . Then  $a = d_4 = 41$ , and  $\phi(n) = n' = 317902032$ .