T-79. 5501
Cryptology

Lecture 3, Feb 5, 2008
-Unicity distance, example
-Extended Euclidean Algorithm
-Chinese remainder theorem
-Euler's totient function
-Euler's theorem
-LFSR

Unicity distance - Example

AES Cryptosystem

$\mathcal{P}$ ={0,1}^{128}, C ={0,1}^{128} , $\mathcal{K}$ ={0,1}^{128}

Data (L) to be encrypted: DES-keys, 64 bits each, each eighth bit is a parity check bit.

$\mathrm{H}_L = 64 - 8 = 56$; $\mathrm{R}_L = 1 - 56/64 = 1/8$

$n_0 \cong \log_2|\mathcal{K}| / \mathrm{R}_L \log_2|\mathcal{P}| = 128/(128/8) = 8$

Prime Numbers

Definition: An integer $p > 1$ is a prime if and only if its only positive integer divisors are 1 and p.

Fact: Any integer $a > 1$ has a unique representation as a product of its prime divisors

$$a = \prod_{i=1}^{t} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$$

where $p_1 < p_2 < \ldots < p_t$ and each e_i is a positive integer.

Some first primes: 2,3,5,7,11,13,17,… For more primes, see:

www.utm.edu/research/**primes**/

Example: Composite (non-prime) numbers and their factorisations:
18 = $2 \cdot 3^2$, 27 = 3^3, 42 = $2 \cdot 3 \cdot 7$, 84773093 = $8887 \cdot 9539$

Euclidean Algorithm

Given two positive integers and their representations as products of prime powers, it would be easy to extract from them the maximum set of common prime powers.

For example gcd(18, 42) = gcd($2 \cdot 3^2$, $2 \cdot 3 \cdot 7$) = $2 \cdot 3 = 6$.

On the other hand, given just one (composite) integer, its factorization is hard to compute (in general).

Euclidean Algorithm is an efficient algorithm for finding the gcd of two integers. It is based on the following fact:

Let $a > b$. Then gcd(a,b) = gcd(a mod b, b).

Example: gcd(42, 18) = gcd(6, 18) = 6.

Example: gcd(595,408) = gcd(187,408) = gcd(187,34) = gcd(17,34) = 17.

Slowest case: Fibonacci sequence 1, 2, 3, 5, 8,13, 21,…,$F_n = F_{n-1} + F_{n-2}$. For example it takes 5 iterations to compute gcd(21,13); in general it takes n-2 iterations to compute gcd(F_n,F_{n-1})

Extended Euclidean Algorithm for integers and computing a modular inverse

Fact: Given two positive integers a and b there exist integers u and v such that

$$u \cdot a + v \cdot b = \gcd(a,b)$$

In particular, if $\gcd(a,b) = 1$, there exist positive integers u and v such that

$$u \cdot a = 1 \pmod{b}, \text{ and } v \cdot b = 1 \pmod{a}.$$

The integers u and v can be computed using the Extended Euclidean Algorithm, which iteratively finds integers r_i , u_i and v_i such that

$$r_0 = b, \;\; r_1 = a; \;\; u_0 = 0, \;\; u_1 = 1; \; v_0 = 1, \; v_1 = 0$$

and for $i = 2,3,\ldots$ we compute q_i such that

$$r_{i-2} = q_i \cdot r_{i-1} + r_i \text{ , where } 0 \leq r_i < r_{i-1} \text{ .}$$

We set: $u_i = u_{i-2} - q_i \cdot u_{i-1}$ and $v_i = v_{i-2} - q_i \cdot v_{i-1}$. Then $r_i = u_i \cdot a + v_i \cdot b$.

Let n be the index for which $r_n > 0$ and $r_{n+1} = 0$. Then

$$r_n = \gcd(a,b) \text{ and } u_n = u \text{ and } v_n = v.$$

Extended Euclidean Algorithm: Example

gcd(595,408) = 17 = $u \cdot 595 + v \cdot 408$

i	q_i	r_i	u_i	v_i
0	-	595	1	0
1	-	408	0	1
2	1	187	1	-1
3	2	34	-2	3
4	5	17	11	-16

Extended Euclidean Algorithm: Examples

$\gcd(595,408) = 17 = 11 \cdot 595 + (-16) \cdot 408$

$= -397 \cdot 595 + 579 \cdot 408$

We get $11 \cdot 595 = 17 \pmod{408}$

and $579 \cdot 408 = 17 \pmod{595}$

If $\gcd(a,b) = 1$, this algorithm gives modular inverses.

Example: $557 \cdot 797 = 1 \pmod{1047}$ that is

$557 = 797^{-1} \pmod{1047}$

If $\gcd(a,b) = 1$, the integers a and b are said to be coprime.

Galois Field

Given a binary polynomial $f(x)$ of degree n, consider a set of binary polynomials with degree less than n. This set has 2^n polynomials. With polynomial arithmetic modulo $f(x)$ this set is a ring.

Fact: If $f(x)$ is irreducible, then this set with 2-ary (binary) polynomial arithmetic is a field denoted by $\mathrm{GF}(2^n)$.

In particular, every nonzero polynomial has a multiplicative inverse modulo $f(x)$. We can compute a multiplicative inverse of a polynomial using the Extended Euclidean Algorithm.

The next slide presents the Extended Euclidean Algorithm for integers. It works exactly the same way for polynomials.

Extended Euclidean Algorithm for polynomials Example

Example: Compute the multiplicative inverse of x^2 modulo x^4+x+1

i	q_i	r_i	u_i	v_i
0		$x^4 + x + 1$	0	1
1		x^2	1	0
2	x^2	$x + 1$	x^2	1
3	x	x	$x^3 + 1$	x
4	1	1	$x^3 + x^2 + 1$	$x + 1$

Extended Euclidean Algorithm for polynomials

Example cont'd

So we get

$$u_4 \cdot x^2 + v_4 \cdot (x^4 + x + 1) = (x^3 + x^2 + 1)x^2 + (x + 1)(x^4 + x + 1) = 1 = r_4$$

from where the multiplicative inverse of x^2 mod $x^4 + x + 1$ is equal to $x^3 + x^2 + 1$.

Chinese Remainder Theorem (two moduli)

Problem: Assume m_1 and m_2 are coprime. Given x_1 and x_2, how to find $0 \le x < m_1 m_2$ such that

$x = x_1 \bmod m_1$

$x = x_2 \bmod m_2$

Solution: Use the Extended Euclidean Algorithm to find u and v such that $u \cdot m_1 + v \cdot m_2 = 1$. Then

$x = x \cdot (u \cdot m_1 + v \cdot m_2) = x \cdot u \cdot m_1 + x \cdot v \cdot m_2$

$= (x_2 + r \cdot m_2) \cdot u \cdot m_1 + (x_1 + s \cdot m_1) \cdot v \cdot m_2.$

It follows that

$x = x \bmod (m_1 m_2) = (x_2 \cdot u \cdot m_1 + x_1 \cdot v \cdot m_2) \bmod (m_1 m_2)$

Chinese Remainder Theorem (general case)

Theorem: Assume $m_1, m_2, \ldots, m_t$ are mutually coprime. Denote $M = m_1 \cdot m_2 \cdot \ldots \cdot m_t$. Given $x_1, x_2, \ldots, x_t$ there exists a unique x, $0 \leq x < M$, such that

$$x = x_1 \bmod m_1$$
$$x = x_2 \bmod m_2$$
$$\ldots$$
$$x = x_t \bmod m_t$$

x can be computed as

$$x = (x_1 \cdot u_1 \cdot M_1 + x_2 \cdot u_2 \cdot M_2 + \ldots + x_t \cdot u_t \cdot M_t) \bmod M,$$

where $M_i = (m_1 \cdot m_2 \cdot \ldots \cdot m_t)/m_i$ and $u_i = M_i^{-1} \pmod{m_i}$.

Proof: For the case $t = 2$ see previous page. The general case was presented on the white boardblackboard, see also textbook.

Chinese Remainder Theorem: Example

Let $m_1 = 7$, $m_2 = 11$, $m_3 = 13$. Then $M = 1001$.

Problem: Compute x, $0 \le x \le 1000$ such that

$x = 5 \bmod 7$

$x = 3 \bmod 11$

$x = 10 \bmod 13$

Solution:

$M_1 = m_2 m_3 = 143$; $M_2 = m_1 m_3 = 91$; $M_3 = m_1 m_2 = 77$

$u_1 = M_1^{-1} \bmod m_1 = 143^{-1} \bmod 7 = 3^{-1} \bmod 7 = 5$; similarly

$u_2 = M_2^{-1} \bmod m_2 = 3^{-1} \bmod 11 = 4$; $u_3 = (-1)^{-1} \bmod 13 = -1$.

Then

$x = (5 \cdot 5 \cdot 143 + 3 \cdot 4 \cdot 91 + 10 \cdot (-1) \cdot 77) \bmod 1001 = 894$

Euler's Totient Function $\phi(n)$

Definition: Let $n > 1$ be integer. Then we set

$$\phi(n) = \#\{\, a \mid 0 < a < n, \gcd(a,n) = 1\},$$

that is, $\phi(n)$ is the number of positive integers less than n which are coprime with n.

For prime p, $\phi(p) = p - 1$. We set $\phi(1) = 1$.

For a prime power p^e, we have $\phi(p^e) = p^{e-1}(p-1)$.

The multiplicative property holds:

$$\phi(m \times n) = \phi(m) \times \phi(n), \text{ for all } m, n, \gcd(m,n) = 1 .$$

Now Euler's totient function can be computed for any integer using its prime factorisation.

Example: $\phi(18) = \phi(2 \times 3^2) = \phi(2) \times \phi(3^2) = (2-1) \times (3-1)3^1 = 6$, that is, the number of invertible (coprime with 18) numbers modulo 18 is equal to 6. They are: $1, 5, 7, 11, 13, 17$.

Euler's Theorem

$$Z_n^* = \{a \mid 0 < a < n, \gcd(a,n) = 1\}, \text{and } \# Z_n^* = \phi(n)$$

Euler's Theorem: For any integers n and a such that $a \neq 0$ and gcd(a,n) = 1 the following holds:

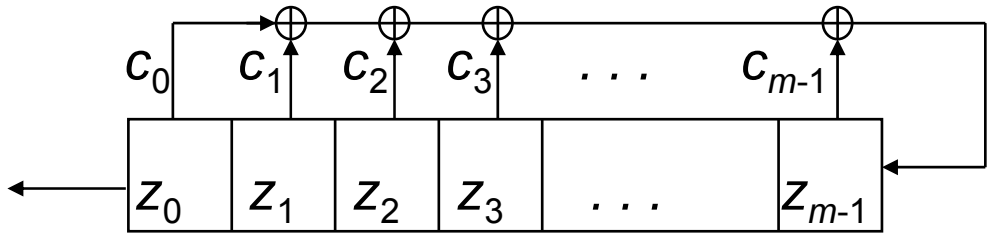
$$a^{\phi(n)} \equiv 1 (\text{mod } n)$$

Fermat's Theorem: For a prime p and any integer a such that $a \neq 0$ and a is not a multiple of p the following holds:

$$a^{p-1} \equiv 1 (\text{mod } p)$$

Linear Feedback Shift Registers

A binary linear feedback shift register (LFSR) is the following device



where the i^{th} tap constant $c_i = 1$, if the switch connected, and $c_i = 0$ if it is open. The contents of the register $z_0, z_1, z_2, z_3, \ldots, z_{m-1}$ are binary values. Given this state of the device the output is z_0 and the new contents are $z_1, z_2, z_3, \ldots, z_{m-1}, z_m$, where z_m is computed using the recursion equation

$$z_m = c_0 z_0 + c_1 z_1 + c_2 z_2 + c_3 z_3 + \ldots + c_{m-1} z_{m-1}$$

The sum is computed *modulo* 2. As this process is iterated, the LFSR outputs a binary sequence $z_0, z_1, z_2, z_3, \ldots, z_{m-1}, z_m, \ldots$ Then the terms of this sequence satisfy the linear recursion relation

LFSR: The first examples

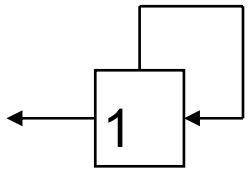
$$z_{k+m} = c_0 z_k + c_1 z_{k+1} + c_2 z_{k+2} + c_3 z_{k+3} + \ldots + c_{m-1} z_{k+m-1}$$

for all $k = 0,1,2,\ldots$

Examples 1.

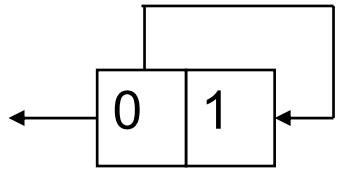
a) $z_i = 0$, $i = 0,1,2,\ldots$ shortest LFSR: ← (no contents, length = 0)

b) $z_i = 1$, $i = 0,1,2,\ldots$ shortest LFSR:



(length $m = 1$)

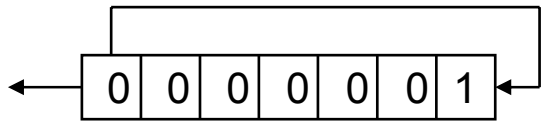
c) sequence 010101… ; shortest LFSR:



(length $m = 2$)

$z_0 = 0$, $z_1 = 1$, $z_{k+2} = z_k$, $k = 0,1,2,\ldots$

d) sequence 00000100000010… LFSR:



LFSR: Connection polynomial

The polynomial over $\mathbf{Z_2}$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_{m-1} x^{m-1} + x^m$$

is called the connection polynomial of the LFSR with taps $c_0\ c_1 \ldots c_{m-1}$.

Given $f(x) = c_0 + c_1 x + \ldots + c_{m-1} x^{m-1} + x^m$, of degree m, we denote by $f^*(x)$ the reciprocal polynomial of f, defined as follows:

$$f^*(x) = x^m f(x^{-1}) = c_0 x^m + c_1 x^{m-1} + c_2 x^{m-2} + \ldots + c_{m-1} x + 1.$$

It has the following properties:

1. deg $f^*(x) \leq$ deg $f(x)$, and deg $f^*(x)$ = deg $f(x)$ if and only if $c_0 = 1$.

2. Let $h(x) = f(x)g(x)$. Then $h^*(x) = f^*(x)g^*(x)$.

The set of sequences generated by the LFSR with connection polynomial $f(x)$ is denoted by $\Omega(f)$:

$$\Omega(f) = \{S = (z_i) \mid z_i \in \mathbf{Z_2};\ z_{k+m} = c_0 z_k + c_1 z_{k+1} + \ldots + c_{m-1} z_{k+m-1},\ k = 0,1,\ldots\}.$$