T-79.5204 **Spring 2007**
Combinatorial Models and Stochastic Algorithms
Tutorial 11, April 19
Problems

1. It can be shown (cf. Brémaud, p. 231) that in a regular Markov chain the expected hitting times μ_{ij} (cf. problem 5 of tutorial 1) can be obtained from the fundamental matrix Z according to the formula $\mu_{ij} = (z_{jj} - z_{ij})/\pi_j$, where π_j denotes the stationary probability of state j. Repeat problem 2 of tutorial 2 using this technique, i.e. compute the expected time to reach state 0 from each of the other states in a simple random walk on the cyclically ordered state set $S = \{0, 1, 2, 3\}$.

2. Establish the validity of the Hastings MCMC design scheme (p. 110 of the lecture notes), i.e. show that with the given choices of acceptance probabilities, the resulting Markov chains are guaranteed to be reversible.

3. Construct a Barker-Hastings sampler for the setting of problem 2 of tutorial 3, with uniform generation probability $q = 1/n$ for each of the Hamming neighbours of a given state $\sigma \in \{0, 1\}^n$. Compare this to the Gibbs sampler designed earlier.

4. (a) Let $A_1, A_2, \ldots$ be a collection of events, and $A = \cap_{n\geq 1} \cup_{m\geq n} A_m$ the event that infinitely many of the A_m occur. Prove the "first Borel-Cantelli lemma", which states that if $\sum_{n\geq 1} \Pr(A_n) < \infty$, then $\Pr(A) = 0$. (*Hint:* $A \subseteq \cup_{m\geq n} A_m$ for all $n \geq 1$.)

   (b) Based on the previous result, prove the following special case of Kolmogorov's Strong Law of Large Numbers: for any sequence $X_1, X_2, \ldots$ of i.i.d. random variables for which $E(X_1) = 0$ and $E(X_1^4) < \infty$, $\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to 0$ almost surely, i.e. denoting $S_n = \sum_{k=1}^{n} X_k$,

$$\Pr(\exists \epsilon > 0 \text{ s.th. } |S_n|/n > \epsilon \text{ infinitely often}) = 0.$$

   (*Hint:* For a given $\epsilon > 0$, consider the events $A_n = \{|S_n| \geq n\epsilon\} = \{S_n^4 \geq (n\epsilon)^4\}$. Apply Markov's inequality and the fact that for independent random variables X_1 and X_2, $E(X_1 X_2) = E(X_1)E(X_2)$.)